Lecture 12

The Standard Model of particle physics

We will discuss the current theory of subatomic world – the Standard Model of particle physics. This theory was formulated in a series of papers by Sh. Glashow, S. Weinberg and A. Salam. By the time the theory was proposed (late 1960s), it was known that there exists an electron (discovered in 1897) and an electron neutrino (1956), a muon (1936) and a muon neutrino (1962). The τ-lepton and the τ-neutrino were unknown.

It was also understood that there is an electromagnetic interaction, facilitated by massless photons, that can be described by a gauge theory known as Quantum Electrodynamics. It was also known that there are weak interactions that cause neutron decay $n \to p + e + \bar{\nu}_e$ and the muon decay $\mu \to e + \bar{\nu}_e + \nu_\mu$. There was Fermi theory, which stipulated that weak decays are described by the following Lagrangian

$$L_F = -\frac{G_F}{\sqrt{2}} \left[\bar{p}\gamma^\mu(1-\gamma_5)n\right] \left[\bar{e}\gamma^\mu(1-\gamma_5)\nu\right] + \text{h.c.}, \tag{1}$$

and a similar one for the muon decay. The Lagrangian Eq. (1) displays maximal parity violation in that only *left-handed fermions* ($\psi_L \sim (1-\gamma_5)\psi$) participate in weak interactions. Also, weak interactions were known to be *short-range*, at variance with electromagnetic interactions.

These two points imply that if weak interactions are to be described by gauge fields, these gauge fields have to couple differently to left- and right-handed fermions and, moreover, these gauge fields have to be massive to make sure that weak interactions are short-range.

The first point – a different role played by left and right fields in weak interactions – has important consequences. Indeed, as we have seen in Lecture 10, a massive electron requires a term $L_m = m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L$ in the Lagrangian. If ψ_L and ψ_R transform differently under gauge transformations, the mass term in the Lagrangian will not be gauge-invariant. A possible way out is to re-use the idea of spontaneous symmetry breaking and apply it to fermions. Indeed, we start by considering a theory with *massless* fermions that couple to a scalar field, e.g. $L_m \to L_Y \sim \bar{\psi}_L\psi_R\varphi + \bar{\psi}_R\psi_L\varphi^\dagger$. The difference with the mass term is that now the field φ can also transform under gauge transformations and it may be possible to adjust quantum numbers of $\psi_{L,R}$ and φ in such a way that L_Y is invariant under gauge transformations. If, however, the field φ undergoes spontaneous symmetry breaking $\varphi \to v$, the Yukawa Lagrangian L_Y produces a mass term for the fermion ψ, i.e. $L_Y \to \bar{\psi}_L\psi_R m + \bar{\psi}_R\psi_L m$. So, similar to how gauge bosons get their masses in the process of spontaneous symmetry breaking, we can set up a gauge invariant theory with *massless* fermions that, after spontaneous symmetry breaking, turns into a theory with massive fermions. This is important since electrons and muons are, in fact, massive.

Let us now discuss how to couple fermions to gauge fields. We have seen in Lecture 10 that if we put a proton and a neutron into an $SU(2)$ doublet, there are terms in the Lagrangian that describe neutron-to-proton transitions. If we look at the Fermi Lagrangian Eq. (1), we see that such transitions are present there. In addition, there are electron-to-(electron) neutrino transitions in the Fermi theory Eq. (1); we can accommodate them by putting the electron and the electron neutrino into an SU(2) doublet. Since the Fermi Lagrangian only contains left-handed fields, we combine the left-handed electron and the left-handed neutrino into an $SU(2)$ (gauge) doublet

$$\Psi_L = \left(\begin{array}{c} e \\ \nu \end{array} \right). \tag{2}$$

Since electrons are massive, we require a right-handed field e_R as well. This field does not participate in weak interactions, i.e. it is not part of the Fermi theory in Eq. (1). However, since QED is parity-conserving, photons do couple to left- and right-handed fields with equal strength. For this reason, we assume that *both* left-handed and right-handed fermions couple to an $U(1)$ field, but we cannot associate this field with the electromagnetic field right away (e.g. neutrinos do not couple to photons). We will also assume that electron neutrinos are massless and for this reason the right-handed neutrino field ν_R is not needed.

Therefore, we consider a theory based on the gauge group $SU(2) \times U(1)$. Both Ψ_L and e_R transform under $U(1)$ but only Ψ_L transforms under $SU(2)$. We will have to break the symmetry, to give masses to (some) gauge bosons and to the electron. We will do this with the help of a scalar complex doublet φ that transforms under both $SU(2)$ and $U(1)$. We will continue with writing down the Lagrangian for such a theory. We will call it $L_{\rm SM}$.

The first term in $L_{\rm SM}$, that is *completely fixed* once the gauge group is specified, is the kinetic term for gauge fields. We write

$$L_{\rm gauge} = L_{SU(2)} + L_{U(1)}, \tag{3}$$

where

$$L_{SU(2)} = -\frac{1}{4} W^i_{\mu\nu} W^{\mu\nu,i}, \qquad L_{U(1)} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{4}$$

with

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ikj} W^k_\mu W^j_\nu. \tag{5}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{6}$$

To move further, we need two covariant derivatives, one for the $SU(2)$ group and the other one for the $U(1)$ group. We have

$$D^{SU(2)}_\mu = \partial_\mu - ig\tau^i W^i_\mu, \tag{7}$$

where $\tau^i = \sigma^i/2$ and

$$D_\mu^{U(1)} = \partial_\mu - ig'\frac{Y}{2}B_\mu, \tag{8}$$

where Y defines a $U(1)$ charge (a "hypercharge") of a particular field in units of the fundamental $U(1)$ gauge coupling g'.

The Lagrangian $L_{\rm gauge}$ describes $3+1$ massless gauge bosons. The theory has to describe weak and electromagnetic interactions that require three massive (weak interactions, charged and neutral currents) bosons and one massless gauge boson (electromagnetism). As we know, this can be achieved by breaking the symmetry spontaneously. To this end, we introduce the Higgs field that is a $SU(2)$ doublet and has a $U(1)$ hypercharge Y_h. We write

$$L_{\rm Higgs}^{\rm kin} = \left(D_\mu\varphi\right)^\dagger\left(D^\mu\varphi\right), \tag{9}$$

where

$$\varphi = \left(\begin{array}{c} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{array}\right) \tag{10}$$

and where

$$D_\mu = \partial_\mu - ig\tau^i W_\mu^i - ig' B_\mu \frac{Y_h}{2}. \tag{11}$$

The part of the Higgs Lagrangian that is responsible for breaking the symmetry is

$$L_{\rm EWSB} = -\frac{\lambda}{4}\left(\varphi^\dagger\varphi - \frac{v^2}{2}\right)^2. \tag{12}$$

The EWSB Lagrangian requires that we choose the non-vanishing vacuum field. We write

$$\varphi(x) = \left(\begin{array}{c} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{array}\right) \tag{13}$$

We know that this is a complete parameterization of the doublet after the symmetry breaking since the rest can be removed by a gauge transformation.

Let us compute the mass spectrum of gauge bosons in such a theory. The mass spectrum follows from $L_{\rm Higgs}^{\rm kin}$ upon substituting $\varphi \to \varphi_{\rm vac}$ there. We find

$$L_{\rm vac}^{\rm kin} \to \varphi_{\rm vac}^T\left[igW_\mu^i\tau^i + ig'B_\mu\frac{Y_h}{2}\right]\left[-igW^{\mu,j}\tau^j - ig'B^\mu\frac{Y_h}{2}\right]\varphi_{\rm vac}, \tag{14}$$

where

$$\varphi_{\rm vac} = \left(\begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array}\right) \tag{15}$$

We expand Eq. (14) and write

$$\varphi_{\rm vac}^T\left[g^2 W_\mu^i W^{j,\mu}\tau^i\tau^j + \frac{Y_h^2}{4}g'^2 B_\mu B^\mu + gg'Y_h W_\mu^i B^\mu\tau^i\right]\varphi_{\rm vac}. \tag{16}$$

Then, we compute

$$
\begin{aligned}
&g^2 W^i_\mu W^{j,\mu}\ \varphi^T_{\rm vac}\tau^i\tau^j\varphi_{\rm vac} = \frac{g^2}{4}W^i_\mu W^{j,\mu}\varphi^T_{\rm vac}\delta_{ij}\hat{1}\varphi_{\rm vac} = \frac{g^2v^2}{8}W^i_\mu W^{i,\mu},\\
&\frac{Y_h^2}{4}g'^2 B_\mu B^\mu \varphi^T_{\rm vac}\varphi_{\rm vac} = \frac{Y_h^2 g'^2 v^2}{8}B_\mu B^\mu,\\
&gg'Y_h W^i_\mu B^\mu \varphi^T_{\rm vac}\tau^i\varphi_{\rm vac} = -\frac{gg'Y_h v^2}{4}W^i_\mu B^\mu \delta^{i3},
\end{aligned}
\tag{17}
$$

so that

$$
\begin{aligned}
L^{\rm kin}_{\rm vac} \to{}& \frac{g^2v^2}{8}\ W^i_\mu W^{i,\mu} + \frac{Y_h^2 g'^2 v^2}{8}B_\mu B^\mu - \frac{gg'Y_h v^2}{4}W^3_\mu B^\mu\\
={}& \frac{v^2g^2}{8}\left(W^1_\mu W^{1,\mu} + W^2_\mu W^{2,\mu}\right)\\
&+\frac{v^2(g^2+Y_h^2g'^2)}{8}\left(\frac{g}{\sqrt{g^2+Y_h^2g'^2}}W^3_\mu - \frac{g'Y_h}{\sqrt{g^2+Y_h^2g'^2}}B_\mu\right)^2.
\end{aligned}
\tag{18}
$$

It follows from Eq. (18) that the two fields W^1, W^2 acquire the mass

$$
m_{1,2} = \frac{gv}{2},
\tag{19}
$$

whereas a combination of W^3 and B fields

$$
Z_\mu = \cos\theta\ W^{(3)}_\mu - \sin\theta\ B_\mu
\tag{20}
$$

acquires the mass

$$
m_Z = \frac{vg}{2\cos\theta}.
\tag{21}
$$

In the above equations, we introduced the so-called weak mixing angle θ; the cosine and sine of this angle is fixed in terms of the gauge coupling and the Higgs boson hypercharge,

$$
\cos\theta = \frac{g}{\sqrt{g^2+Y_h^2g'^2}}, \qquad \sin\theta = \frac{g'Y_h}{\sqrt{g^2+Y_h^2g'^2}}.
\tag{22}
$$

A combination of fields that is orthogonal to Eq. (20) reads

$$
A_\mu = \sin\theta\ W^{(3)}_\mu + \cos\theta\ B_\mu.
\tag{23}
$$

An important consequence of Eq. (18) is that the field A_μ remains massless. We would like to associate it with the photon.

We continue with the discussion on leptons. We will only consider an electron and an electron neutrino since muons and muon neutrinos are included into the SM in an identical way. As we already said, the left-handed fields are $SU(2)$ doublets and the right-handed electrons are $SU(2)$ singlets. We consider massless neutrinos and, therefore, we do not introduce the right-handed neutrino field. We write the Lagrangian

$$L_{\rm F} = \bar{\psi}_L i\hat{D}_L \psi_L + \bar{e}_R i\hat{D}_R e_R, \tag{24}$$

where

$$\begin{aligned} D_L^\mu &= \partial^\mu - igW_\mu^i \tau^i - ig' B_\mu \frac{Y_L}{2}, \\ D_R^\mu &= \partial^\mu - ig' B_\mu \frac{Y_R}{2}. \end{aligned} \tag{25}$$

As we already discussed in Lecture 10, we cannot write the mass term for an electron since it mixes left- and right-handed fields and these fields transform in a different way under $SU(2)$ and $U(1)$ gauge groups. We will return to this question after we study how gauge bosons interact with fermions.

To understand this, we neglect partial derivatives in Eq. (25) and consider only terms of the type $\bar{\psi} V_\mu \gamma^\mu \psi = \bar{\psi} \hat{V} \psi$, where V_μ is a gauge field. Then

$$\begin{aligned} \bar{\psi}_L i\hat{D}_L \psi_L &\to \frac{1}{2}\Big[\bar{\nu}_L (g\hat{W}^3 + g'\hat{B}Y_L)\nu_L + \bar{e}_L(-g\hat{W}^3 + g'\hat{B}Y_L)e_L \\ &\qquad + g\bar{\nu}_L(\hat{W}^1 + i\hat{W}^2)e_L + g\bar{e}_L(\hat{W}^1 - i\hat{W}^2)\nu_L\Big], \\ \bar{e}_R i\hat{D}_R e_R &\to \frac{g'}{2} Y_R \bar{e}_R \hat{B} e_R. \end{aligned} \tag{26}$$

To understand how electrons and neutrinos interact with gauge fields, we express W_μ^3 and B_μ through mass eigenstates of gauge fields. To this end we write

$$\begin{aligned} W_\mu^3 &= \cos\theta Z_\mu + \sin\theta A_\mu, \\ B_\mu &= -\sin\theta Z_\mu + \cos\theta A_\mu. \end{aligned} \tag{27}$$

We use these equations to determine couplings of Z_μ and A_μ to electrons and neutrinos. Of particular importance to us is the coupling of the photon field A_μ to fermions. This is so because we know how this coupling should look like. In particular, the photon should *not* couple to neutrinos since they have no electric charge and it should couple to both left- and right-handed electrons with equal force that is proportional to the electric charge of the electron. These features provide important constraints for the theory.

We begin with neutrino's couplings to photons. This coupling will come from the $\bar{\nu}_L \cdots \nu_L$ term in Eq. (26) if we replace both W_μ^3 and B_μ with A_μ following Eq. (27). We find

$$\frac{1}{2}\bar{\nu}_L(g\hat{W}^3 + g'\hat{B}Y_L)\nu_L \to \frac{1}{2}(g\sin\theta + g'Y_L\cos\theta)\bar{\nu}_L\hat{A}\nu_L = \frac{gg'\,(Y_H + Y_L)}{\sqrt{g^2 + g'^2 Y_L^2}}\bar{\nu}_L\hat{A}\nu_L. \tag{28}$$

Hence, to ensure that photons do not couple to neutrinos, we need to choose hypercharges of the Higgs boson and the left-handed lepton doublet to satisfy the following equation

$$Y_H + Y_L = 0. \tag{29}$$

The second constraint is that photons couple to left and right-handed electrons with equal strength (no parity-violation in QED). The electrons appear in Eq. (26) in two places, as left- and right handed. Again, we replace W^3 and B with A in a way compatible with Eq. (27) and obtain

$$\begin{aligned} &\frac{1}{2}\bar{e}_L(-g\hat{W}^3 + g'\hat{B}Y_L)e_L + \frac{g'}{2}Y_R\bar{e}_R B e_R \\ &\to \frac{1}{2}(g'Y_L\cos\theta - g\sin\theta)\bar{e}_L\hat{A}e_L + \frac{g'}{2}Y_R\cos\theta\bar{e}_R\hat{A}e_R \\ &= \frac{gg'\,(Y_L - Y_H)}{2\sqrt{g^2 + g'^2 Y_L^2}}\bar{e}_L\hat{A}e_L + \frac{gg'Y_R}{2\sqrt{g^2 + g'^2 Y_L^2}}\bar{e}_R\hat{A}e_R. \end{aligned} \tag{30}$$

To ensure that left- and right-handed fermions coupled to A_μ with the strength proportional to the electric charge, the following equations have to be satisfied

$$Y_L - Y_H = Y_R, \qquad \frac{gg'Y_R}{2\sqrt{g^2 + g'^2 Y_L^2}} = q_e. \tag{31}$$

The first equation is the requirement that there is no parity violation in the electron coupling to photons; the second requirement is that photons couple to electrons with a strength defined by an electron charge q_e.

One can use the above considerations to write the couplings of electrons and neutrinos to Z's and A's. However, we will not pursue this direction here; instead, we will discuss how to make sure that the electron has the right mass. To this end, we need to write a gauge-invariant coupling between left-handed and right-handed fermions and the Higgs fields

$$L_Y = f_e\left[\bar{\psi}_L\varphi e_R + h.c.\right], \tag{32}$$

where we assumed the Yukawa coupling f_e to be real. The Yukawa Lagrangian in Eq. (32) is obviously invariant under $SU(2)$ transformations since *both*, the left-handed field and the Higgs field are $SU(2)$ doublets. However, if we perform the $U(1)$ transformation each field transforms in a way that involves individual hypercharges, i.e.

$$\varphi \to e^{i\frac{Y_H}{2}\theta(x)}\varphi, \qquad \psi_L \to e^{i\frac{Y_L}{2}\theta(x)}\psi_L, \qquad e_R \to e^{i\frac{Y_R}{2}\theta(x)}e_R. \tag{33}$$

Hence, the Yukawa Lagrangian is invariant under $U(1)$ transformations if the following equation

$$Y_H + Y_R - Y_L = 0 \tag{34}$$

holds true. It is easy to check that Eqs. (29), Eq. (31) and Eq. (34) are all compatible with each other provided that hypercharges are chosen to satisfy the following equations

$$Y_H = -Y_L, \qquad\qquad Y_R = 2Y_L. \tag{35}$$

We then choose $Y_L = 1$ and obtain

$$Y_H = -1, \qquad\qquad Y_R = 2, \qquad\qquad Y_L = 1. \tag{36}$$

It is then easy to see that the Yukawa Lagrangian Eq. (32) provides a mass to electrons after the symmetry breaking. Indeed, upon replacing the Higgs field with its vacuum expectation value we obtain

$$L_Y \to f_e \left[\bar{\varphi}_L \varphi_{\rm vac} e_R + h.c.\right] = \frac{f_e v}{\sqrt{2}} \left[\bar{e}_L e_R + h.c.\right] = m_e \left[\bar{e}_L e_R + h.c.\right], \tag{37}$$

where $m_e = f_e v/\sqrt{2}$ is the electron mass. As we see, neutrino remains massless.

We note that our theory describes interactions of photons, Z-bosons, $W^{\pm}$-bosons and the Higgs boson between themselves and with electrons and neutrinos. Electrons are massive and Higgs interactions with electrons are proportional to the Yukawa coupling f_e which is given by the ratio of the electron mass and the vacuum expectation value of the Higgs field. To make our theory complete, we need to introduce the muon and the muon neutrino and τ-lepton and τ-neutrino as well as quarks. The additional leptons are put in by simply repeating what we did for the electron and the electron neutrino. The situation with quarks is slightly more complicated since different quark generations can "mix" with each other. This mixing is described by the so-called Cabibbo-Kobayashi-Maskawa matrix and leads to the phenomenon of CP-violation.