Lecture 11

Spin-statistics connection

When identical particles are discussed in Quantum Mechanics courses, we state that wave functions of bosons should be symmetric while wave functions of fermions should be anti-symmetric. This has important observable implications both in quantum physics itself and in e.g. statistical physics. Why this requirement should be true remains a bit of a miracle. This isn't surprising since it really takes quantum field theory to explain the connection between spin and statistics; in quantum field theory, there is a theorem that states that consistent relativistic quantum field theory is only possible if bosonic fields are quantized with commutators and fermionic fields with anti-commutators. The properties of wave functions then follow automatically.

We will try to illustrate the spin-statistics connection by quantizing the fermion field *in the wrong way*, which is the only natural thing to do if we do not know any better. So lets try. The Dirac action is

$$S_D = \int \mathrm{d}^4x \, \bar{\psi} \left(i\hat{\partial} - m \right) \psi. \tag{1}$$

The canonical momentum is

$$\pi = \frac{\delta S_D}{\delta \partial_t \psi} = \bar{\psi} i \gamma_0 = i\psi^\dagger. \tag{2}$$

We can now quantize the Dirac theory by imposing the canonical quantization condition on the canonical momentum and the field

$$[\pi_a(t,\vec{x}), \psi_b(t,\vec{y})] = -i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \quad \to \qquad [\psi_a^\dagger(t,\vec{x}), \psi_b(t,\vec{y})] = -\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}), \tag{3}$$

where a, b are the spinor indices. We assume that ψ commutes with ψ and $\psi^\dagger$ commutes with $\psi^\dagger$ at equal times.

Similar to what we have done for the scalar field, we derive the Hamiltonian of the theory using the standard relation between the Lagrangian and the Hamiltonian

$$H = \int \mathrm{d}^3\vec{x} \, (\pi \partial_t \psi - \mathcal{L}) = \int \mathrm{d}^3\vec{x} \psi^\dagger(t,\vec{x})\gamma_0 \left(-i\vec{\gamma}\vec{\nabla} + m \right) \psi(t,\vec{x}). \tag{4}$$

To determine the equation of motion for ψ, we write

$$i\partial_t \psi = [\psi, H]. \tag{5}$$

We use Eq. (3) to compute the commutator and obtain

$$\left(i\hat{\partial} - m \right) \psi = 0, \tag{6}$$

which is the Dirac equation that the operator ψ should satisfy.

Similar to the scalar case, we then write the operator ψ as a linear combination of the solutions of the Dirac equation

$$\psi(t,\vec{x}) = \sum_{\lambda=1,2} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left[u_\lambda(\vec{k}) a_{\vec{k},\lambda} e^{-i\omega_k t + i\vec{k}\vec{x}} + v_\lambda(\vec{k}) b_{\vec{k},\lambda} e^{i\omega_k t - i\vec{k}\vec{x}} \right]. \tag{7}$$

The sum over λ is the sum over the possible polarizations of a fermion. Since ψ is a complex field, we require different creation and annihilation operators for positive- and negative-energy solutions. The spinors u and v are solutions of the following Dirac equations

$$(\hat{k} - m) u_\lambda(\vec{k}) = 0, \quad (\hat{k} + m) v_\lambda(\vec{k}) = 0. \tag{8}$$

From Eq. (7) it is straightforward to obtain $\psi^\dagger(t,\vec{x})$. We find

$$\psi^\dagger(t,\vec{x}) = \sum_{\lambda=1,2} \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \left[u_\lambda^\dagger(\vec{k}) a^\dagger_{\vec{k},\lambda} e^{i\omega_k t - i\vec{k}\vec{x}} + v_\lambda^\dagger(\vec{k}) b^\dagger_{\vec{k},\lambda} e^{-i\omega_k t + i\vec{k}\vec{x}} \right]. \tag{9}$$

We now need to check that the commutation relations in Eq. (3) can be fulfilled if we impose regular commutation relations

$$[a_{\vec{k}_1,\lambda_1}, a^\dagger_{\vec{k}_2,\lambda_2}] = \delta_{\lambda_1\lambda_2}(2\pi)^3\delta^{(3)}(\vec{k}_1 - \vec{k}_2), \qquad [b_{\vec{k}_1,\lambda_1}, b^\dagger_{\vec{k}_2,\lambda_2}] = \delta_{\lambda_1\lambda_2}(2\pi)^3\delta^{(3)}(\vec{k}_1 - \vec{k}_2). \tag{10}$$

on creation and annihilation operators. All the other commutators are assumed to vanish. We find

$$\begin{aligned}
[\psi_a(t,\vec{x}), \psi_b^\dagger(t,\vec{y})] = & \sum_{\lambda_1,\lambda_2} \int\int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3\sqrt{2\omega_{k_1}}} \frac{\mathrm{d}^3\vec{k}_2}{(2\pi)^3\sqrt{2\omega_{k_2}}} \\
& \times \Bigg[[a_{\vec{k}_1,\lambda_1}, a^\dagger_{\vec{k}_2,\lambda_2}] u_{a,\lambda_1}(\vec{k}_1) u^\dagger_{b,\lambda_2}(\vec{k}_2) e^{-ik_1x + ik_2y} \\
& + [b_{\vec{k}_1,\lambda_1}, b^\dagger_{\vec{k}_2,\lambda_2}] v_{a,\lambda_1}(\vec{k}_1) v^\dagger_{b,\lambda_2}(\vec{k}_2) e^{ik_1x - ik_2y} \Bigg],
\end{aligned} \tag{11}$$

where we introduced $k_1x = \omega_{k_1}t - \vec{k}_1\vec{x}$ and $k_2y = \omega_{k_2}t - \vec{k}_2\vec{y}$. We now use the commutation relations Eq. (10) and find

$$\begin{aligned}
[\psi_a(t,\vec{x}), \psi_b^\dagger(t,\vec{y})] = & \sum_{\lambda_1} \int \frac{\mathrm{d}^3\vec{k}_1}{(2\pi)^3 2\omega_{k_1}} \Bigg[u_{a,\lambda_1}(\vec{k}_1) u^\dagger_{b,\lambda_1}(\vec{k}_1) e^{i\vec{k}_1(\vec{x}-\vec{y})} \\
& + v_{a,\lambda_1}(\vec{k}_1) v^\dagger_{b,\lambda_1}(\vec{k}_1) e^{-i\vec{k}_1(\vec{x}-\vec{y})} \Bigg].
\end{aligned} \tag{12}$$

To proceed further, we require the following sums

$$\begin{aligned}
\sum_\lambda u_{a,\lambda}(\vec{k}_1) u^\dagger_{b,\lambda}(\vec{k}_1) &= \Big[(\omega_k\gamma_0 - \vec{k}\vec{\gamma} + m)\gamma_0 \Big]_{ab} \\
\sum_\lambda v_{a,\lambda}(\vec{k}_1) v^\dagger_{b,\lambda}(\vec{k}_1) &= \Big[(\omega_k\gamma_0 - \vec{k}\vec{\gamma} - m)\gamma_0 \Big]_{ab},
\end{aligned} \tag{13}$$

that can be checked using explicit solutions of the Dirac equation. We substitute Eq. (13) into Eq. (12) and change $\vec{k}_1 \to -\vec{k}_1$ in the second term on the right hand side. We then find

$$[\psi_a(t,\vec{x}), \psi_b^\dagger(t,\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\delta_{ab}, \tag{14}$$

which is indeed the canonical commutation relation Eq. (3).

Having satisfied the commutation relations for canonical variables, we can construct the Hamiltonian and determine the Hilbert space. The Hamiltonian operator is given in Eq. (4). It is straightforward to express it in terms of creation and annihilation operators. We find

$$H = \sum_\lambda \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \omega_k \left[a^\dagger_{\lambda,\vec{k}} a_{\lambda,\vec{k}} - b^\dagger_{\lambda,\vec{k}} b_{\lambda,\vec{k}} \right]. \tag{15}$$

We see that our theory is pathological in that there is no ground state; in fact, by creating more and more b-quanta, we produce states with more and more negative energy. Note that renaming $b \to b^\dagger$ in Eq. (7) would not have worked either since in this case the canonical commutation relations would not be satisfied.

The resolution of this problem can be obtained if we quantize the theory by imposing conditions on the *anti-commutator* of the field ψ and its conjugate

$$\{\psi^\dagger(t,\vec{x}), \psi(t,\vec{y})\} = -\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}). \tag{16}$$

This leads to the anti-commutation of creation and annihilation operators

$$\{a_{\vec{k}_1,\lambda_1}, a^\dagger_{\vec{k}_2,\lambda_2}\} = \delta_{\lambda_1\lambda_2}(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2), \qquad \{b_{\vec{k}_1,\lambda_1}, b^\dagger_{\vec{k}_2,\lambda_2}\} = \delta_{\lambda_1\lambda_2}(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2), \tag{17}$$

as well as $\{a,b\} = \{b,a\} = 0$ etc. These conditions ensure that the wave function of a state with several fermions is automatically anti-symmetric. For example,

$$|\vec{k}_1,\vec{k}_2\rangle = \sqrt{2\omega_{k_1}}\sqrt{2\omega_{k_2}} a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle = -\sqrt{2\omega_{k_1}}\sqrt{2\omega_{k_2}} a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_1}|0\rangle = -|\vec{k}_2,\vec{k}_1\rangle. \tag{18}$$