Lecture 10

The Dirac Lagrangian

So far in these lectures we talked about (elementary) bosons. However, almost all matter around us is made out of elementary *fermions*, i.e. particles with spin 1/2. As you know from the course in Quantum Mechanics, fermions with spin 1/2 are described by the Dirac equation. It reads

$$\left(i\hat{\partial} - m\right)\psi = 0, \tag{1}$$

where $\hat{\partial} = \partial_\mu \gamma^\mu$. The four 4×4 matrices γ^μ are known as Dirac matrices. The function ψ is a four-component complex vector that we call a "spinor".

Dirac matrices satisfy the following equation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{2}$$

so that $(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $i = 1, 2, 3$.

There are different (equivalent) ways to represent Dirac matrices, that are connected to each other by unitary transformations. If the Dirac representation is chosen, they read

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{3}$$

where σ^i, $i = 1, 2, 3$ are Pauli matrices.

In Quantum Mechanics, we interpreted ψ as a fermion (electron) wave function. We would like to promote it to a "field" and write a Lagrangian that produces the Dirac equation. We write the action as

$$S_D = \int \mathrm{d}^4 x \, \bar{\psi} \left[i\hat{\partial} - m\right] \psi. \tag{4}$$

The field $\bar{\psi}$ is the Dirac-conjugate spinor. It reads

$$\bar{\psi} = \psi^\dagger \gamma^0. \tag{5}$$

Since ψ is a complex field, it and its complex (or Dirac) conjugate field can be considered independent. Hence, to find the extremum of S_D, we need to vary $\bar{\psi}$ and ψ separately. We find

$$\begin{aligned} \delta S_D &= \int \mathrm{d}^4 x \delta\bar{\psi} \left[i\hat{\partial} - m\right] \psi + \int \mathrm{d}^4 x \bar{\psi} \left[i\hat{\partial} - m\right] \delta\psi \\ &= \int \mathrm{d}^4 x \delta\bar{\psi} \left[i\hat{\partial} - m\right] \psi + \int \mathrm{d}^4 x \bar{\psi} \left[-i\overleftarrow{\hat{\partial}} - m\right] \delta\psi, \end{aligned} \tag{6}$$

where the arrow in the last term indicates that the derivative acts on $\bar{\psi}$.

Requiring that $\delta S_D = 0$, for arbitrary $\delta\psi$ and $\delta\bar{\psi}$, we obtain two equations

$$\left[i\hat{\partial} - m\right]\psi = 0, \quad \bar{\psi}\left[-i\overleftarrow{\hat{\partial}} - m\right] = 0. \tag{7}$$

To check that they are consistent, we take the first one and perform hermitian conjugation. We find

$$0 = \left(\left[i\hat{\partial} - m\right]\psi\right)^\dagger = \psi^\dagger\left[-i\overleftarrow{\partial^\mu}\gamma_\mu^\dagger - m\right] \tag{8}$$

To simplify this further, we note that hermitian-conjugated Dirac matrices satisfy the following equations

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i. \tag{9}$$

It follows

$$\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0, \tag{10}$$

that can be checked using their explicit form in Eq. (3). Using this equation and the fact that $\gamma_0^2 = 1$, we write

$$0 = \psi^\dagger\left[-i\overleftarrow{\partial^\mu}\gamma_\mu^\dagger - m\right] = \psi^\dagger\left[-i\overleftarrow{\partial^\mu}\gamma_0\gamma_\mu\gamma_0 - m\gamma_0\gamma_0\right] = \psi^\dagger\gamma_0\left[-i\overleftarrow{\partial^\mu}\gamma_\mu - m\right]\gamma_0. \tag{11}$$

The equation for $\bar{\psi}$ obviously follows.

Another interesting point is that the presence of the Dirac-conjugate spinor in the Dirac action is essential for this action being real. We find

$$\begin{aligned} S_D^* = S_D^\dagger &= \int \mathrm{d}^4x\ \psi^\dagger\left[-i\overleftarrow{\partial^\mu}\gamma_\mu^\dagger - m\right]\gamma_0^\dagger\psi = \int \mathrm{d}^4x\ \psi^\dagger\left[i\partial^\mu\gamma_\mu^\dagger - m\right]\gamma_0^\dagger\psi \\ &= \int \mathrm{d}^4x\ \psi^\dagger\gamma_0\gamma_0\left[i\partial^\mu\gamma_\mu^\dagger - m\right]\gamma_0\psi = \int \mathrm{d}^4x\ \bar{\psi}\left[i\partial^\mu\gamma_\mu - m\right]\psi = S_D. \end{aligned} \tag{12}$$

Hence, the use of the Dirac conjugate spinor in the action makes the action real. S_D is also a Lorentz-scalar; this can be shown by studying how ψ and $\bar{\psi}$ change under Lorentz transformations.

We would like to understand how to couple the fermion fields to other fields that we discussed in the previous lectures. We will start with a simple remark that, typically, ψ is a complex field. Therefore, we can express the action in terms of another field ψ' that differs from ψ by a phase

$$\psi = e^{i\alpha}\psi'. \tag{13}$$

The Dirac action is obviously invariant under this transformation

$$S_D[\bar{\psi}, \psi] = S_D[\bar{\psi}', \psi']. \tag{14}$$

We have seen that there is a conserved current that can be associated with such symmetry transformations. We write

$$J^\mu = \frac{\delta L_D}{\delta\partial_\mu\psi}\Delta\psi + \Delta\bar{\psi}\frac{\delta L_D}{\delta\partial_\mu\bar{\psi}} = \frac{\delta L_D}{\delta\partial_\mu\psi}\Delta\psi = \bar{\psi}i\gamma^\mu\Delta\psi \to \bar{\psi}i\gamma^\mu\psi, \tag{15}$$

where we have used (c.f. Eq. (13)) $\Delta\psi = \psi$ and $\delta L_D/\delta\partial_\mu\bar{\psi} = 0$. Hence,

$$J^\mu = \bar{\psi}\gamma^\mu\psi \tag{16}$$

and

$$\partial_\mu J^\mu = 0. \tag{17}$$

We have seen that one can force the theory to be invariant under the local version of phase transformations in Eq. (13) by introducing a gauge field. In this particular case, we write

$$\partial_\mu \to D_\mu = \partial_\mu - igA_\mu, \tag{18}$$

and, similar to the case of a scalar complex field, we consider the transformations of ψ and A_μ

$$\psi \to e^{i\alpha(x)}\psi, \quad A_\mu \to A_\mu + \frac{\partial_\mu\alpha}{g}, \tag{19}$$

that do not change the action. Hence,

$$S_D[\bar{\psi}, \psi, A_\mu] = \int \mathrm{d}^4x\ \bar{\psi}\left[i\hat{D} - m\right]\psi. \tag{20}$$

The equation of motion is

$$[i\hat{D} - m]\psi = 0, \tag{21}$$

which looks identical to the Dirac equation in the external electromagnetic field.

It is clear that it should be possible to extend the above discussion to the non-abelian gauge fields. Of course, we will need to have more than a single Dirac field to make this happen. So, let us consider a concrete example. It was realized early on that protons and neutrons are very similar in that they have very similar masses and that it makes very little difference for strong interaction processes if a proton or a neutron participates in a given reaction. This similarity became known as the "isospin symmetry". We can formalize it by combining the (four-component) spinors for the proton (ψ_p) and the neutron (ψ_n) into a "double-spinor" (Ψ) as follows

$$\Psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}. \tag{22}$$

We can write an action for the spinor Ψ as

$$S = \int \mathrm{d}^4x\bar{\Psi}\begin{pmatrix} i\hat{\partial} - m & 0 \\ 0 & i\hat{\partial} - m \end{pmatrix}\Psi = \int \mathrm{d}^4x\bar{\Psi}\left(i\hat{\partial} - m\right)\Psi, \tag{23}$$

where in the last step we simplified the notation in that we decided not to show a 2×2 identity matrix anymore.

It is clear that the action Eq. (23) is invariant under the following transformation

$$\Psi = U\Psi', \tag{24}$$

where U is any 2×2 unitary matrix. This is quite obvious since, from the point of view of "flavor indices", the action S depends on $\Psi^\dagger\Psi$ that does not change if the transformation Eq. (24) is applied.

The consequences of Eq. (24) are interesting. For example, taking

$$U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{25}$$

we find

$$\psi'_p = i\psi_n, \quad \psi'_n = i\psi_p, \tag{26}$$

i.e. what used to be a proton became a neutron (up to a phase) and what used to be a neutron became a proton (up to a phase). The symmetry obviously doesn't care about us giving names to these "identical" particles.

We may be unhappy with having to choose a "proton" and a "neutron" globally and may require that the true action S should be made invariant under *local* choices, i.e. an x-dependent transformation as in Eq. (24) should not change the action. This is done in exactly the same way as before in that we promote derivatives to covariant derivatives. We write

$$\hat{\partial} \to \hat{D} = \hat{\partial} - ig\hat{A}, \tag{27}$$

where

$$\hat{A} = \gamma^\mu \sum_{a=1}^{3} A_\mu^{(a)} \tau^a, \tag{28}$$

and τ_a are the three generators of the $SU(2)$ Lie algebra. The action reads

$$S = \int \mathrm{d}^4x \bar{\Psi} \left\{ \begin{pmatrix} i\hat{\partial} - m & 0 \\ 0 & i\hat{\partial} - m \end{pmatrix} + g \sum_{a=1}^{3} A_\mu^{(a)} \gamma^\mu \tau^{(a)} \right\} \Psi. \tag{29}$$

The last term describes interactions of gauge fields with fermions. It is instructive to write it explicitely. We find

$$\sum_{a=1}^{3} A_\mu^{(a)} \gamma^\mu \tau^{(a)} = \frac{1}{2} \begin{pmatrix} A_\mu^{(3)}\gamma^\mu & (A_\mu^{(1)} - iA_\mu^{(2)})\gamma^\mu \\ (A_\mu^{(1)} + iA_\mu^{(2)})\gamma^\mu & -A_\mu^{(3)}\gamma^\mu \end{pmatrix}, \tag{30}$$

so that

$$\begin{aligned} g\bar{\Psi} \sum_{a=1}^{3} A_\mu^{(a)} \gamma^\mu \tau^{(a)} \Psi &= \frac{g}{2} \left[A_\mu^{(3)} \left(\bar{\psi}_p \gamma^\mu \psi_p - \bar{\psi}_n \gamma^\mu \psi_n \right) \right] \\ &+ \frac{g}{2} \left[(A_\mu^{(1)} - iA_\mu^{(2)}) \bar{\psi}_p \gamma^\mu \psi_n + (A_\mu^{(1)} + iA_\mu^{(2)}) \bar{\psi}_n \gamma^\mu \psi_p \right]. \end{aligned} \tag{31}$$

We see that if the proton and the neutron are considered to be part of the same multiplet, gauge interactions cause transitions between them. For example, if we denote $(A_\mu^{(1)} + iA_\mu^{(2)})/\sqrt{2} = A_\mu^\dagger$ and consider $A_\mu^\dagger$ to be a new field, we see that Eq. (31) contains terms

$A^\dagger_\mu \bar{\psi}_n \gamma^\mu \psi_p$ that describe a process where a proton becomes a neutron by emitting a gauge boson $A^\dagger_\mu$.

We will use this observation to construct the fermion sector of the Standard Model.

For the next step, we need to discuss some properties of very energetic fermions. There is an interesting way to split the Dirac equation into two coupled equations. It requires us to introduce yet another Dirac matrix γ_5,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{32}$$

and, as a simple calculation shows, it reads

$$\gamma_5 = \left(\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right). \tag{33}$$

This matrix has the following properties

$$\gamma_5^2 = 1, \quad \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0. \tag{34}$$

We can use these properties to introduce two projection operators

$$P_L = \frac{1-\gamma_5}{2}, \quad P_R = \frac{1+\gamma_5}{2}. \tag{35}$$

We call them projection operators because they have the following properties

$$P_L + P_R = 1, \quad P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0. \tag{36}$$

We use these operators to write

$$\psi = P_L\psi + P_R\psi = \psi_L + \psi_R. \tag{37}$$

Now, going back to the Dirac equation, we write

$$\left(i\hat{\partial} - m\right)\psi = 0 \;\; \Rightarrow \;\; \left(i\hat{\partial} - m\right)\psi_L + \left(i\hat{\partial} - m\right)\psi_R = 0. \tag{38}$$

Now, multiply the last equation with P_R from the left, use

$$P_L\gamma_\mu = \gamma_\mu P_R, \tag{39}$$

combine is with Eq. (36) and find

$$i\hat{\partial}\psi_L = m\psi_R, \tag{40}$$

A similar computation gives

$$i\hat{\partial}\psi_R = m\psi_L. \tag{41}$$

It is interesting that equations for ψ_R and ψ_L decouple in the massless ($m \to 0$) limit. Also, since $\partial_\mu\psi \sim p_\mu\psi \sim E\psi$, the admixture of ψ_R into ψ_L and the admixture of ψ_L into ψ_R is proportional to m/E and is suppressed for ultra-relativistic fermions.

It is straightforward to check that the action can be written in the following way

$$S_D = \int \mathrm{d}^4x \left[\bar{\psi}_L i\hat{\partial}\psi_L + \bar{\psi}_R i\hat{\partial}\psi_R - m\bar{\psi}_L\psi_R - m\bar{\psi}_R\psi_L\right]. \tag{42}$$

We see again that if we set $m \to 0$, the ψ_L and ψ_R fields decouple. Conversely, the mass term mixes left and right fields; this point will be essential for the construction of the Standard Model Lagrangian.

The left- and right-handed Dirac fields are fully described by one two-component spinor. Indeed, if

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{43}$$

where ϕ and χ are two-component spinors, then

$$\psi_L = \begin{pmatrix} \phi - \chi \\ -(\phi - \chi) \end{pmatrix}, \quad \psi_R = \begin{pmatrix} \phi + \chi \\ (\phi + \chi) \end{pmatrix}, \tag{44}$$

Hence, to specify $\psi_{L,R}$ one needs to specify *one* two-component spinor and not two two-component spinors as in the case of the original Dirac fermion ψ.

To better understand the meaning of ψ_L and ψ_R, it is instructive to perform a *parity transformation*

$$x^\mu = (x_0, \vec{x}) \to x'^\mu = (x_0, -\vec{x}). \tag{45}$$

We make this change in the Dirac equation and obtain

$$\left[i\left(\gamma^0 \frac{\partial}{\partial x_0} - \vec{\gamma}\frac{\partial}{\partial \vec{x'}}\right) - m\right]\psi(x'_0, -\vec{x'}) = 0. \tag{46}$$

Multiply this equation with the matrix γ_0 from the left and move it all the way to the function ψ. Since γ_0 anti-commutes with $\vec{\gamma}$, we find

$$\left[i\left(\gamma^0 \frac{\partial}{\partial x_0} + \vec{\gamma}\frac{\partial}{\partial \vec{x'}}\right) - m\right]\gamma_0\psi(x'_0, -\vec{x'}) = \left[i\partial'_\mu\gamma^\mu - m\right]\gamma_0\psi(x'_0, -\vec{x'}) = 0. \tag{47}$$

It follows from the last equation that

$$\gamma_0\psi(x) = \psi'(x'), \tag{48}$$

where $\psi'(x')$ is the solution of the 'parity-transformed' Dirac equation. If we write $\psi = \psi_L + \psi_R$, we find

$$\psi'_L(x) = \gamma_0\psi_R(x), \quad \psi'_R(x) = \gamma_0\psi_L(x). \tag{49}$$

This means, that under parity transformation, ψ_R becomes ψ_L and vice versa.

The electromagnetic interactions conserve parity. This means that left and right components of the spinor field ψ_L and ψ_R have identical interactions with the electromagnetic field. It was believed that parity conservation was the feature of all interaction until 1950's when maximal parity violation was observed in weak interactions. We will explore this observation to construct the Standard Model of particle physics.