Problem 18: The $\beta$-function for $\alpha_s$

The value of the strong coupling $\alpha_s$ depends on the renormalization scale $\mu$. The dependence is determined through differential equation

$$ \frac{d\alpha_s(\mu)}{d(ln\mu)} = \beta(\alpha_s(\mu)) = -\frac{\beta_0}{2\pi}\alpha_s(\mu)^2 - \frac{\beta_1}{4\pi^2}\alpha_s(\mu)^3 - \ldots. $$

The coefficients $\beta_0$, $\beta_1$ etc. can be calculated:

$$ \beta_0 = 11 - \frac{2}{3}n_f, \quad \beta_1 = 51 - \frac{19}{3}n_f, $$

where $n_f$ is the number of flavors.

a) Assume that the $\alpha_s$ is known at the scale $\mu = Q$. Calculate $\alpha_s(\mu)$ for $\mu \neq Q$ to first order (i.e. neglect the term with $\beta_1$).

b) Show that one can write $\alpha_s(\mu)$ as

$$ \alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)}. $$

Express the $\Lambda$ through $\alpha_s(Q)$. How does $\Lambda$ depend on $n_f$? What happens for $\mu \approx \Lambda$?

c) Now take into account the term with $\beta_1$. Show that, to a leading order in $\ln(\mu^2/\Lambda^2)^{-1}$, the following holds:

$$ \alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda^2)} \left( 1 - \frac{2\beta_1 \ln(\ln(\mu^2/\Lambda^2))}{\beta_0^2 \ln(\mu^2/\Lambda^2)} \right) + O(1/\ln^3(\mu^2/\Lambda^2)) $$

with

$$ \Lambda = Q e^{-2\pi/\beta_0\alpha_s(Q)} \left( \frac{4\pi + 2\beta_1 \alpha_s(Q) / \beta_0}{\beta_0 \alpha_s(Q)} \right)^{\beta_1 / \beta_0^2}. $$