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Sheet 9

Problem 14: Scheme dependence in QCD β-function

The relationship between bare QCD coupling $g^{(0)}$ and trenormalized coupling $\bar{g}$ in $\overline{\text{MS}}$-scheme is given by

$$g^{(0)} = \bar{Z}\bar{g}\mu^{\varepsilon}\frac{e^{\varepsilon\gamma_{\mathrm{E}}}}{(4\pi)^{\varepsilon/2}} \quad ,$$

where μ is renormalization scale and $\bar{Z}$ is the renormalization constant. Consider, in addition, another mass-independent scheme S with renormalized coupling $\hat{g}$ and renormalization constant $\hat{Z}$:

$$g^{(0)} = \hat{Z}\hat{g}\mu^{\varepsilon}\frac{e^{\varepsilon\gamma_{\mathrm{E}}}}{(4\pi)^{\varepsilon/2}} \quad .$$

The two schemes differ by a finite renormalization:

$$\bar{Z} = z(\bar{g})\hat{Z} \quad \text{mit} \quad z(\bar{g}) = 1 + z_0\bar{g}^2 + z_1\bar{g}^4 + z_2\bar{g}^6 + \ldots \quad .$$

a) Express $\hat{g}$ in terms of $\bar{g}$.

b) The coupling in $\overline{\text{MS}}$-scheme fulfils renormalization group equation

$$\mu\frac{d\bar{g}(\mu)}{d\mu} = \bar{\beta}(\bar{g}(\mu)) = -\bar{\beta}_0\bar{g}^3 - \bar{\beta}_1\bar{g}^5 - \bar{\beta}_2\bar{g}^7 - \ldots \quad .$$

Analogous equation in scheme S is

$$\mu\frac{d\hat{g}(\mu)}{d\mu} = \hat{\beta}(\hat{g}(\mu)) = -\hat{\beta}_0\hat{g}^3 - \hat{\beta}_1\hat{g}^5 - \hat{\beta}_2\hat{g}^7 - \ldots \quad .$$

Express $\bar{\beta}(\bar{g})$ in terms of $\hat{\beta}$, z and $\bar{g}$. Show that $\hat{\beta}_0 = \bar{\beta}_0$ and $\hat{\beta}_1 = \bar{\beta}_1$ and express $\hat{\beta}_2$ in terms of $\bar{\beta}_2$.