Institut für Theoretische Teilchenphysik

Einführung in die Flavourphysik
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Exercise Sheet 5

Problem 8: Pions and isospin

We denote by $(q_u, q_d) \equiv (u, d)$ the isospi-quark doublet of the first generation and consider the following field operators:

$$\pi_{ij}(\mathbf{p}) \equiv \int d^3\mathbf{x}\, e^{-i\mathbf{p}\mathbf{x}} {:}\bar{q}_i(0,\mathbf{x})\gamma_5 q_j(0,\mathbf{x}){:} \quad \text{with } i,j \in \{u,d\},$$

where $:\cdot:$ denotes the normal ordering (all creation operators are to the left of all annihilation operators). Pion states are created by acting with appropriate linear combinations of the operators $\pi_{ij}(\mathbf{p})$ on the vacuum $|0\rangle$.

a) The quark fields transform under charge conjugation C, parity transformation P and Poincaré transformations $U(\Lambda, a)$ as following:

$$\mathsf{C}q_i(x)\mathsf{C}^\dagger = i\gamma^0\gamma^2\bar{q}_i^\mathsf{T}(x) \equiv \mathcal{C}\bar{q}_i^\mathsf{T}(x) \quad , \quad \mathsf{P}q_i(t,\mathbf{x})\mathsf{P}^\dagger = \gamma^0 q_i(t,-\mathbf{x}) \quad ,$$
$$U(\Lambda,a)q_i(x)U^\dagger(\Lambda,a) = S(\Lambda^{-1})q_i(\Lambda x + a) \quad .$$

The $S(\Lambda)$ is the representation of the Lorentz transformation Λ in the space of Dirac spinors:

$$S(\mathbb{1}+\omega) = \exp(-\tfrac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}) \quad \text{with} \quad \sigma^{\mu\nu} = \tfrac{i}{4}[\gamma^\mu, \gamma^\nu] \quad .$$

How do the operators $\pi_{ij}(\mathbf{p})$ transform under space translations, charge conjugation and parity transformations? How do $\pi_{ij}(\mathbf{0})$ transform under rotations? What are the momentum, angular momentum and parity quantum number of the state $\pi_{ij}(\mathbf{0})|0\rangle$? Hint: You can first show that $\mathsf{C}\bar{q}_i\mathsf{C}^\dagger = q_i^\mathsf{T}\mathcal{C}$ and $\mathsf{P}\bar{q}_i(t,\mathbf{x})\mathsf{P}^\dagger = \bar{q}_i(t,-\mathbf{x})\gamma^0$.

b) If the masses of up- and down quarks are set equal the QCD has the additional SU(2)-isospin symmetry. The quark fields transform under isospin transformations in the fundamental representation:

$$q_i(x) \to U_{ij}q_j(x)\,,$$

with $U \in \mathbb{C}^{2\times 2}$, $U^\dagger = U^{-1}$ and $\det U = 1$. How do the operators $\pi_{ij}(\mathbf{p})$ transform under isospin? Define $\tilde{\pi}_{ij}(\mathbf{p}) = -\varepsilon_{ik}\pi_{kj}(\mathbf{p})$, where $\varepsilon = i\sigma_2$ is two-dimensional Levi-Civita tensor. How do the operators $\tilde{\pi}_{ij}(\mathbf{p})$ transform under isospin transformations? Decompose the vector space in the irreducible representation of SU(2) in terms of four operators $\tilde{\pi}_{ij}(\mathbf{p})$, i.e. form the linear combinations (analogous to angular momentum addition in quantum mechanics) $\pi^{I,I_3}(\mathbf{p})$ of the operators $\tilde{\pi}_{ij}(\mathbf{p})$ that transform under the spin-I- representation of the $SU(2)$-isospin. Determine the transformation behavior of these operators under charge conjugation and parity transformations and their electric charges. Hint: For every SU(2)-Matrix, $U\ \varepsilon U^* = U\varepsilon$.

c) Pion states with momentum $\mathbf{p}$ are created by acting on the vacuum with operators $\pi^{I,I_3}(\mathbf{p})$ with $I = 1$ and $I_3 \in \{-1, 0, +1\}$. We denote these operators by $\pi^{\pm}(\mathbf{p})$ and $\pi^0(\mathbf{p})$. The G-parity transformation is defined as the product of charge-conjugation C and a specific isospin transformation:

$$Gq_i(x)G^\dagger = (e^{i\pi\sigma_2/2})_{ij}\mathsf{C}q_j(x)\mathsf{C}^{-1} \quad ,$$

where σ_2 is the second Pauli matrix. How do the operators $\pi^{\pm}$ and π^0 transform under G-parity? Does the QCD allow for the scattering processes as $\pi^+\pi^- \to \pi^+\pi^-\pi^0$?