Classical Theoretical Physics II

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Exercise Sheet 9

Issue: 15.06.18 – Submission: 22.06.18 before 09:30 – Discussion: 26.06.18

Exercise 1: Tri-atomic molecule **10 points**

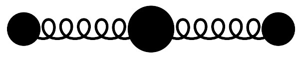

Figure 1

In this exercise we will model a linear tri-atomic molecule, with a central atom with mass M, and two identical atoms with mass m at the ends. A real-world example of such a molecule is CO_2. The same example was discussed in the lectures in a slightly different way.
We will model the atomic forces between the central atom and the side atoms, as springs with spring constant k that are at equilibrium when the distance between the atoms are b. We will only consider movement along the axis of the molecule, making it a one-dimensional problem. Denoting the position of the central atom x_2, and of the side atoms x_1 and x_3, the potential energy is therefore

$$U = \frac{k}{2}\Big((x_2 - x_1 - b)^2 + (x_3 - x_2 - b)^2\Big)$$

(a) What is the Lagrangian of the system?

(b) We now do a variable change to a new set of coordinates η that are relative to the equilibrium position. If x_{20} is the position of the central atom at some specific time (so $\dot{x}_{20} = 0$), these coordinates are $x_2 = x_{20} + \eta_2$, $x_1 = x_{20} + \eta_1 - b$, $x_3 = x_{20} + \eta_3 + b$. What is the Lagrangian in the new coordinates?

(c) The equation of motion for η_1 is $m\ddot{\eta}_1 + k(\eta_1 - \eta_2) = 0$. What are the EOMs for η_2 and η_3?

(d) Let us assume a sinusoidal form for the solutions, such that $\ddot{\eta}_i = -\omega^2\eta_i$, with ω being an oscillation frequency. Show that such an ansatz allows the EOM system to be expressed on matrix form, as $A\vec{\eta} = 0$, where $\vec{\eta} = (\eta_1, \eta_2, \eta_3)$ and

$$A = \begin{bmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{bmatrix}$$

(e) Such an equation system only allows for non-zero solutions if $\det(A) = 0$. Why is that?

(f) Show that only three different (non-negative) values of ω allow for non-zero solutions: $\omega = 0$, $\omega = \sqrt{\frac{k}{m}}$, and $\omega = \sqrt{\frac{k}{m}\big(1 + \frac{2m}{M}\big)}$.

(g) In the lectures we solved the same problem by finding the eigenvectors and eigenvalues of the matrix

$$B = \begin{bmatrix} k/m & -k/m & 0 \\ -k/M & 2k/M & -k/M \\ 0 & -k/m & k/m \end{bmatrix}$$

The eigenvectors were $(1, 1, 1)$, $(1, 0, -1)$, and $(1, -2m/M, 1)$, and the eigenvalues were the squares of three values of ω found above. Why are these two approaches to solving the system equivalent?

(h) What is the physical interpretation of the three oscillation modes?

Exercise 2: Four beads on a ring **10 points**

Four beads of mass m numbered from one to four, are mounted on a ring with radius r along which they are allowed to slide. The particles are joined with springs, which have the property that the potential energy of a spring connecting two beads at angles θ_i and θ_j, is $\frac{1}{2}kr^2(\theta_j - \theta_i)^2$.

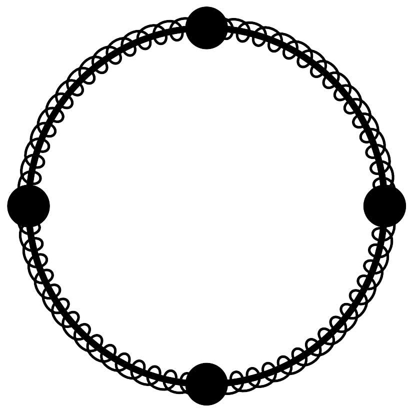

Figure 2

(a) An equilibrium position for the system is $\theta_1 = 0$, $\theta_2 = \pi/2$, $\theta_3 = \pi$, $\theta_4 = 3\pi/2$. Changing to coordinates relative to these equilibrium positions: $\theta_1 = \phi_1$, $\theta_2 = \phi_2 + \pi/2$, $\theta_3 = \phi_3 + \pi$, $\theta_4 = \phi_4 + 3\pi/2$, write down the Lagrangian and the equations of motion for the beads.

(b) We will now consider small oscillations around that equilibrium position. Argue that an ansatz consisting of oscillations with (angular) frequency ω, allows for the system of EOMs to be written as

$$A\vec{\phi} = \vec{0} \qquad \text{with} \qquad A = -\omega^2 I + B \qquad \text{and} \qquad \vec{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4) \quad (1)$$

where I is the 4×4 unit matrix.

(c) Find the matrix B as a function of m and k.

(d) Guess an orthogonal set of four eigenvectors of the matrix B and insert them in eq. (1) to extract the corresponding eigenfrequencies ω.
Hint 1: When guessing the eigenvectors consider the symmetries of the problem. If one bead moves in a certain way, how should the other beads move for the movement to be stable?
Hint 2: The (positive) eigenfrequencies ω should be 0, $\sqrt{2k/m}$, $\sqrt{2k/m}$, and $2\sqrt{k/m}$.

(e) What is the physical interpretation of these four oscillation modes?

(f) Let U be the 4×4 matrix that has as its four rows, an orthonormal (i.e. orthogonal and with length 1) set of eigenvectors of B. Introduce coordinates ψ, defined by $\vec{\psi} = U\vec{\phi}$, and write the Lagrangian and the equations of motion for these new coordinates.

(g) What makes the ψ coordinates good? How comes it turns out like that?