Classical Theoretical Physics II

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Exercise Sheet 8

Issue: 08.06.18 – Submission: 15.06.18 before 09:30 – Discussion: 19.06.18

Exercise 1: Oscillations on a circle — 5 points

In this problem we will consider the setup depicted in Figure 1. A particle with mass m and electrical charge q is free to move along the circle with radius r. It is affected by a gravitational force of size mg pointing downward, and by a repulsive Coulomb force from the other particle (with charge q) that is fixed at the bottom of the circle. We use units such that the Coulomb potential between two particles of charge q a distance d apart, is q^2/d.

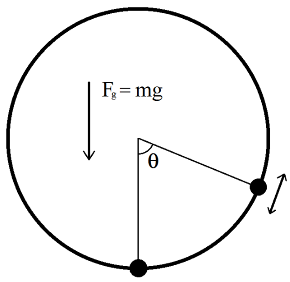

Figure 1

(a) Show that the Lagrangian for the particle may be given by

$$L = \tfrac{1}{2}mr^2\dot{\theta}^2 + mgr\cos(\theta) - \frac{q^2}{2r\sin(\theta/2)}\,. \tag{1}$$

(b) Assume that the values of m and q allow for an equilibrium angle between $\theta = 0$ and $\theta = \pi$. Find that equilibrium angle as a function of r, g, m and q.

(c) The movement may be described as small oscillations around the equilibrium position. Find the (angular) frequency ω of those oscillations.

Exercise 2: Damped oscillations — 4 points

In this exercise we will consider damped oscillations in one dimension. Damped oscillations are the result of movement in the force field

$$F = -kx - \mu\dot{x}\,.$$

The movement obeys the equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0\,. \tag{1}$$

(a) For a particle with mass m, what are γ and ω_0 in terms of k, μ, and m?

(b) Whenever $\gamma < 2\omega_0$, the oscillations are called weakly damped. Show that for that case, a solution to the equations of motion is

$$x = x_0 e^{-\gamma t/2}\cos(\omega_d t)\,, \tag{2}$$

with

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2/4}\,. \tag{3}$$

Hint: It is enough to check that the solution is valid.

(c) What is the ratio between the amplitudes of successive swings to the same side?

(d) When friction is present, the energy is not conserved. What is the energy of the weakly damped oscillator as a function of time? (Assume that $\gamma \ll \omega_0$)

Exercise 3: Forced oscillations **7 points**

In this exercise we will consider the consequences of applying an external force F to a harmonic oscillator. This is known as forced oscillations.

(a) Forced oscillations are a solution to the equation

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}\,. \tag{1}$$

As we have seen in the lectures, a particular solution is

$$x_p(t) = \int_{t_0}^{t} \frac{F(\tau)}{m\omega} \sin(\omega(t-\tau))d\tau\,, \tag{2}$$

where t_0 may be any constant value. This means that the general solution is given as

$$x(t) = A\sin(\omega t + \phi) + x_p(t)\,, \tag{3}$$

with A and ϕ being free parameters. Verify the validity of this solution, by direct insertion of eq. (3) into eq. (1).

(b) One commonly encountered case is a harmonic force $F(t) = F_0 \sin(\Omega t)$ with Ω in general being different from the eigenfreqency ω. Derive the general expression for $x(t)$ for this harmonic force. You may put $t_0 = 0$.

(c) The expression derived in question (b), reduces to

$$x(t) = \frac{F_0/m}{\omega^2 - \Omega^2} \sin(\Omega t) \tag{4}$$

for certain values of A and ϕ. Which ones?

(d) Any periodic, odd function $f(x)$ with period $2\pi/\Omega$, can be written as a so-called Fourier series

$$f(t) = \sum_{n=1}^{\infty} c_n \sin(\Omega n t)\,, \tag{5}$$

where the c_n are given as

$$c_n = \frac{\Omega}{\pi} \int_{-\pi/\Omega}^{\pi/\Omega} f(t)\, \sin(\Omega n t)\, dt \tag{6}$$

Given eq. (5), prove eq. (6).
Hint: $\int_{-\pi}^{\pi} \sin(nz)\sin(\nu z)dz = \delta_{n\nu}\pi$ for $\{n, \nu\}$ being integers.

(e) Let us consider the function $F(t)$ shown in Figure 2. Between $t = -\pi/\Omega$ and $t = \pi/\Omega$ it is given as $f(t) = \Omega F_z t/\pi$, and outside that interval it is repeated periodically. Find an expression as a Fourier series, for the oscillations induced by applying a force described by this function, to the harmonic oscillator (with eigenfrequency ω and mass m).
Hint: Use the Fourier series, the fact that the differential equation is linear, and equation (4).

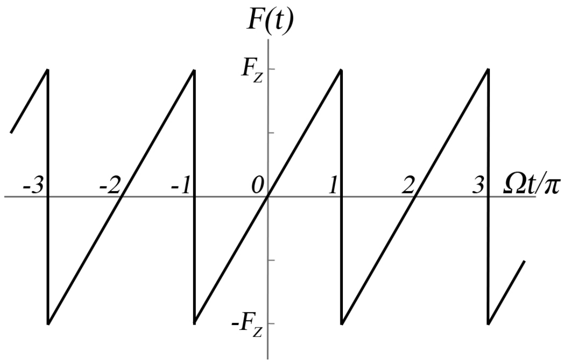

Figure 2: The driving function considered in Exercise 3, question (e).

Exercise 4: Induced oscillations — 4 points

A light particle of mass m, may oscillate as a harmonic oscillator with eigenfrequency ω. A much heavier particle moves along a straight line with velocity v, and passes the light particle at a minimum distance ρ, at time $t = 0$. The heavy particle acts upon the light particle with a force, that forces the oscillations. That force can be approximated as $F = f_0 \exp(-\alpha d^2)$, where d is the distance between the object and the heavy particle.

(a) The equation of motion for the light particle, is

$$\ddot{x} + \omega^2 x = \frac{F(t)}{m}\,, \tag{1}$$

with

$$F(t) = F_0 \exp(-\beta t^2)\,. \tag{2}$$

What are F_0 and β as functions of f_0, α, v, and ρ?

(b) Derive the function $x(t)$ describing the oscillations of the light particle long after the heavy particle has passed ($t = +\infty$), under the assumption that the light particle is at rest initially ($t = -\infty$).