Classical Theoretical Physics II

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Exercise Sheet 7

Issue: 01.06.18 – Submission: 08.06.18 before 09:30 – Discussion: 12.06.18

Exercise 1: Scattering off a surface — 5 points

In the lectures we have calculated the cross section for scattering off a solid sphere. In this problem we consider elastic scattering of particles, coming from $z = +\infty$ with initial velocities parallel to the z-axis, off a solid object consisting of the points

$$\left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; \sqrt{x^2+y^2} \le b\sin\left(\tfrac{z}{a}\right) \text{ and } 0 \le z \le \pi a\right\} . \tag{1}$$

In other words, the edge of the object is the surface of revolution around the z-axis generated by

$$\rho(z) = b\sin\left(\frac{z}{a}\right) \quad \text{for} \quad 0 \le z \le \pi a . \tag{2}$$

The parameters a and b are positive.

(a) Make a sketch of this surface. Draw the path of a particle with impact parameter $0 < \rho < b$ and indicate the deflection angle χ. Demonstrate from the sketch that half of the deflection angle is related to the slope of the surface,

$$\tan\left(\frac{\chi}{2}\right) = \frac{d\rho(z)}{dz} . \tag{3}$$

(b) Derive from eq. (3) the relation between the impact parameter and deflection angle,

$$\rho(\chi) = \sqrt{b^2 - a^2\tan^2\left(\frac{\chi}{2}\right)} . \tag{4}$$

(c) Calculate the differential cross section

$$\frac{d\sigma}{d\chi} = 2\pi\rho(\chi)\left|\frac{d\rho(\chi)}{d\chi}\right| . \tag{5}$$

(d) What are the minimal and maximal values of the scattering angle, χ_{min} and χ_{max}, corresponding to impact parameters $\rho \to b$ and $\rho \to 0$, respectively?

(e) Compute the total cross section

$$\sigma = \int_{\chi_{\text{min}}}^{\chi_{\text{max}}} d\chi\, \frac{d\sigma}{d\chi} . \tag{6}$$

Can you explain the simple result for σ from a geometrical point of view?

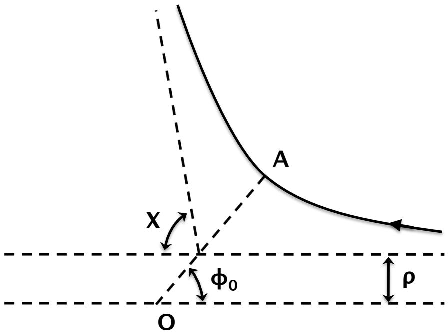

Figure 1: Definition of the impact parameter ρ, scattering angle χ and angle ϕ_0 in Exercise 2.

Exercise 2: Scattering in $1/r^2$ potential **5 points**

In the lectures we have discussed scattering in a Coulomb potential $U(r) = \pm\alpha/r$ and derived the famous Rutherford formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\chi/2)} \; , \tag{1}$$

where $d\Omega = 2\pi \sin(\chi) d\chi$. In this problem we will compute the cross section for scattering in a more strongly localised repulsive potential,

$$U(r) = \alpha/r^2 \quad \text{with} \quad \alpha > 0 \; . \tag{2}$$

The scattering angle is denoted by $\chi = |\pi - 2\phi_0|$ with ϕ_0 given by

$$\phi_0 = \int_{r_{\min}}^{\infty} dr \, \frac{\rho/r^2}{\sqrt{1 - \rho^2/r^2 - U(r)/E}} \; . \tag{3}$$

The lower bound of integration $r_{\min}$ is equal to the distance between the origin and the turning point A, see Figure 1.

(a) Give the defining equation for $r_{\min}$. Show that in this case $r_{\min} = \sqrt{\rho^2 + \alpha/E}$.

(b) Perform the integral in eq. (3) and show that a particle with energy E and impact parameter ρ is deflected by an angle

$$\chi = \pi \left(1 - \frac{1}{\sqrt{1 + \alpha/(E\rho^2)}}\right) \; . \tag{4}$$

(c) Invert eq. (4) to obtain $\rho = \rho(\chi)$. Show that the differential scattering cross section, defined in eq. (5) of the previous exercise, evaluates to

$$\frac{d\sigma}{d\chi} = \frac{2\pi^3\alpha}{E} \frac{\pi - \chi}{\chi^2(2\pi - \chi)^2} \; . \tag{5}$$

(d) Sketch the differential cross section as a function of χ between $\chi = 0$ and $\chi = \pi$. How does the function behave as $\chi \to 0$? Is the total cross section finite, or divergent?

Exercise 3: Particle capture **10 points**

In this problem we are interested in the process where a particle comes from far away and goes toward the origin $r = 0$ as $t \to \infty$. In other words, the particle is captured by the central potential.

(a) The characteristic aspect of this problem is that there is *no* turning point. Show that this implies that the energy E must be larger than the effective potential $U_{\text{eff}}(r) = U(r) + E\rho^2/r^2$:

$$E - U_{\text{eff}}(r) > 0 \quad \text{for all } r \ . \tag{1}$$

(b) For eq. (1) to hold, $U_{\text{eff}}(r)$ must in particular be finite as $r \to 0$. Show that particle capture can therefore only happen if $U(r)$ tends to $-\infty$ as $-\beta/r^2$ with $\beta > E\rho^2$, or if $U(r)$ is proportional to $-1/r^n$ with $n > 2$.

Consider an attractive potential $U(r) = -\beta/r^2$ with $\beta > 0$.

(c) Calculate the effective potential $U_{\text{eff}}(r)$ and sketch it for the two cases $\beta > E\rho^2$ and $\beta < E\rho^2$.

(d) What are the minimal and maximal values of the impact parameter for which particle capture takes place? Calculate the total cross section for particle capture

$$\sigma = \int d\sigma = \int d\rho\, 2\pi\rho \ . \tag{2}$$

Consider now the potential $U(r) = \alpha/r - \beta/r^2$ with $\alpha > 0$ and $\beta > E\rho^2$.

(e) Sketch the effective potential $U_{\text{eff}}(r)$ in this case.

(f) Calculate the maximum value U_0 of the effective potential.

(g) For a particle to reach the center, its energy must certainly exceed U_0. Show that this gives the following upper bound on the impact parameter:

$$\rho^2 < \frac{\beta}{E} - \frac{\alpha^2}{4E^2} \ . \tag{3}$$

(h) Using the fact that the impact parameter is positive, derive that

$$E > \frac{\alpha^2}{4\beta} \ , \tag{4}$$

in order for a particle to reach the center.

(i) Show that the total cross section for particle capture is given by

$$\sigma = \begin{cases} \pi\left(\frac{\beta}{E} - \frac{\alpha^2}{4E^2}\right) & \text{if } E > \frac{\alpha^2}{4\beta} \ , \\ 0 & \text{if } E < \frac{\alpha^2}{4\beta} \ . \end{cases} \tag{5}$$