Classical Theoretical Physics II

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Exercise Sheet 6

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Note: On this exercise sheet we will use $\vec{L}$ as the angular momentum vector, and $l = |\vec{L}|$ as the size of the angular momentum. M will denote the mass of the sun.

Exercise 1: Precession of the perihelion **10 points**

Einsteins' general theory of relativity predicts that the orbits of the planets are not stable ellipses as predicted by Newton, but rather ellipses for which the perihelion (i.e. the point on the orbit closest to the sun) slowly rotates around the sun. The correct prediction of the size of this effect for the orbit of the planet Mercury was the first triumph of general relativity, and helped establishing that theory as the successor to Newtonian gravitation[1].
The effect of the relativistic corrections, can be modeled as an additional term in the gravitational potential which decreases as r^{-2}, such that the full potential is

$$U(r) = -\frac{k}{r} + \frac{C}{r^2}\,, \tag{1}$$

where k is the same as in Newtonian gravity, and C is the relativistic correction.

(a) Write down the Lagrangian for an object with mass m moving in this potential.

(b) What are the conserved quantities? Explain why this implies that the motion takes place on a plane that is naturally described by polar coordinates r and θ.

(c) For an object moving in a central force field, θ as a function of r can be written as

$$\theta = \int \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mU(r)}{l^2} - \frac{1}{r^2}}}\,, \tag{2}$$

where l is the angular momentum of the object (around the origin), and E is its energy. Show that this implies that the shape of the orbit is

$$r = \frac{\rho(1-\epsilon^2)}{1+\epsilon\cos(\alpha\theta)}\,, \tag{3}$$

where ρ, ϵ, and α are functions of E, l, m, k, and C.
Hint 1: Follow the same steps as the derivation of the Kepler orbit presented in the lectures.

[1] https://en.wikipedia.org/wiki/Tests_of_general_relativity

Hint 2: Use the integral

$$\int \frac{dz}{\sqrt{az^2+bz+c}} = \frac{1}{\sqrt{-a}} \arccos\left(-\frac{b+2az}{\sqrt{b^2-4ac}}\right), \tag{4}$$

and the variable change $z = 1/r$.

(d) The expression for α is

$$\alpha = \sqrt{1+\frac{2mC}{l^2}}\,. \tag{5}$$

What are the expressions for ρ and ϵ?

(e) Make a sketch of the orbit for values of α close to one. Show on the sketch α, ρ, and ϵ. What is the essential difference between the cases of $\alpha = 1$ and $\alpha \neq 1$.

(f) Assume in the following that C is much smaller than other scales in the problem.
Derive that the "rate of the precession of the perihelion", i.e. the change in the angular position of the perihelion for each orbit of the planet, is

$$\Delta\theta = \frac{-2\pi mC}{l^2}\,. \tag{6}$$

(g) Einsteins' theory predicts that

$$C = \frac{-3G^2M^2m}{c^2}\,, \tag{7}$$

where G is Newtons' gravitational constant, M is the mass of the sun, and c the speed of light.
Substituting the observables ϵ and ρ for the not directly measurable quantities E and l, show that the rate of the precession of the perihelion may be expressed as

$$\Delta\theta = \frac{6\pi MG}{c^2(1-\epsilon^2)\rho}\,. \tag{8}$$

Hint: You may use the Newtonian expressions for k, ρ, and ϵ:

$$k = GMm\,, \qquad \rho = -\frac{GMm}{2E}\,, \qquad \epsilon = \sqrt{1+\frac{2El^2}{mk^2}}\,. \tag{9}$$

(h) The observed value of the precession of the perihelion of the planet Mercury is 5600 arcseconds per century (an arcsecond or "second of arc" is 1/3600 of a degree), but most of this effect is due gravitational effects from the other planets, well described by Newtonian physics. How big is the relativistic contribution to the precession of the perihelion of Mercury?

Hint: The planet Mercury has $\epsilon = 0.206$, $\rho = 57.9 \times 10^9$m, orbital period $\tau = 88.0$ (Earth) days, and $m = 3.30 \times 10^{23}$kg. Additionally $M = 1.99 \times 10^{30}$kg, $G = 6.67 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2}$, and $c = 3.00 \times 10^8 \frac{\mathrm{m}}{\mathrm{s}}$.

Exercise 2: The Runge-Lenz vector **10 points**

In a previous problem set, we saw that the Runge-Lenz vector $\vec{A}$ is conserved in the case of the Kepler problem.

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r} , \tag{1}$$

where $\vec{p}$ is the momentum vector, $\vec{L}$ is the angular momentum vector, $k = GMm$, and $\hat{r} = \vec{r}/|r|$ is the unit vector in the radial direction.

(a) Make a sketch of an elliptic orbit, and indicate on it the direction of the Runge-Lenz vector at various locations in the orbit.

(b) Let us denote the angle between $\vec{A}$ and $\vec{r}$ as θ, such that $\vec{A} \cdot \vec{r} = |A||r| \cos(\theta)$ By comparing this with an explicit calculation of $\vec{A} \cdot \vec{r}$, derive the expression

$$r = \frac{l^2}{mk + A\cos\theta} , \tag{2}$$

where $l = |\vec{L}|$.

(c) By comparing this with the expression for the shape of an elliptic orbit,

$$r = \frac{(1-\epsilon^2)\rho}{1+\epsilon\cos\theta} , \tag{3}$$

express A in terms of the eccentricity ϵ.

(d) For deviations from the $1/r$ potential, the Runge-Lenz vector is no longer conserved.
Show that the time-derivative of the Runge-Lenz vector for a general radial potential $U(r) = -k/r + \delta U(r)$, is

$$\frac{d\vec{A}}{dt} = f(r)\hat{r} \times \vec{L} , \tag{4}$$

where $f(r) = -\frac{d\,\delta U(r)}{dr}$.
Hint: Use the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$.

(e) Use angular momentum conservation to re-express the time-derivative as an angular derivative, and obtain the expression

$$\frac{d\vec{A}}{d\theta} = -f(r)mr^2\hat{L} \times \hat{r} , \tag{5}$$

where $\hat{L} = \vec{L}/l$ is a unit vector in the $\vec{L}$ direction.

(f) Let us now re-examine the $1/r^2$ perturbation from the previous exercise, such that $f(r) = \frac{2C}{r^3}$. Use this to approximate the change of the Runge-Lenz vector over one orbit, as

$$\Delta\vec{A} = \frac{-2\pi Cm}{l^2}\hat{L} \times \vec{A} . \tag{6}$$

Hint 1: Parametrize the space such that $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$ and $\hat{L} = \hat{z}$.
Hint 2: $\int_0^{2\pi} \cos\theta \sin\theta = 0$ and $\int_0^{2\pi} \cos^2\theta = \pi$.

(g) Compare this to the expression for the "rate of the precession of the perihelion" from the previous exercise:

$$\Delta\theta = \frac{-2\pi mC}{l^2}\,. \tag{7}$$

What is the interpretation of this? How does it fit with the sketch made in the first question?