Classical Theoretical Physics II

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Exercise Sheet 5

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Exercise 1: EOM for various potentials — 5 points

A particle with mass m is moving in a one-dimensional potential $U(x)$.

(a) Write up the Lagrangian for the particle.

(b) The Lagrangian is independent of time. What quantity is conserved in that case?

(c) In that case, the time may be written as a function of the position as

$$t = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dy}{\sqrt{E - U(y)}}\,, \tag{1}$$

as we have seen in the lectures. Find x as a function of time for the potential $U(x) = \alpha/x^2$, assuming positive α and positive energy.

(d) Repeat the previous question for the potential $U(x) = \alpha x^2$ (with positive α and positive energy).

Hint: Useful integrals for this exercise:

$$\int \frac{dz}{\sqrt{a - bz^2}} = \frac{1}{\sqrt{b}} \arccos\left(\frac{\sqrt{b}\,z}{\sqrt{a}}\right), \tag{2}$$

$$\int \frac{dz}{\sqrt{a + b/(z^2)}} = \frac{z\sqrt{a + b/(z^2)}}{a}. \tag{3}$$

Exercise 2: Potential with local maximum — 8 points

In this exercise we will consider a particle in the potential shown in fig. 1.

(a) Argue that the movement of the particle will consist of oscillations between x_0 and x_1.

(b) Model the potential close to $x = 0$ as $U(x) = U(0) - \frac{1}{2}\kappa x^2$, and consider the particle moving from left to right. How long does it spend between $x = -\delta$ and $x = \delta$? (δ is a positive number, significantly smaller than $L = x_1 - x_0$). Hint: Use the integral

$$\int_{-c}^{c} \frac{dz}{\sqrt{a + bz^2}} = \frac{1}{\sqrt{b}} \left(\log\left(a + 2bc^2 + 2c\sqrt{b(a + bc^2)}\right) - \log a\right). \tag{1}$$

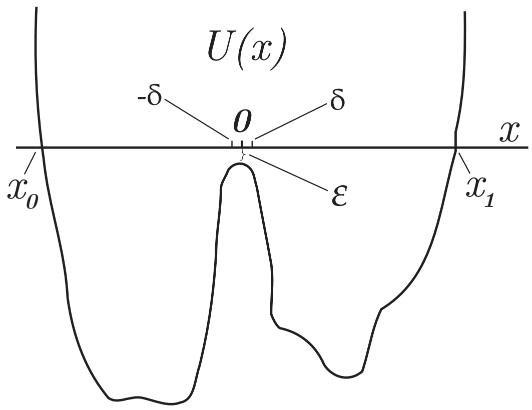

Figure 1: This is a sketch of the potential discussed in exercise 2. The horizontal axis shows the position, and the vertical axis the value E of the energy. The width of the potential is $L = x_1 - x_0$.

(c) Let us consider the case where $\epsilon = E - U(0)$ is much smaller than all other scales in the problem. Show that the time spent between $x = -\delta$ and $x = \delta$ (where δ is much smaller than the width of the potential L) can be approximated as

$$t_1 = \sqrt{\frac{m}{\kappa}} \log\left(\frac{2\delta^2\kappa}{\epsilon}\right). \tag{2}$$

(d) Show that when x is not between $-\delta$ and δ, the speed of the particle fulfills

$$|v| > \delta\sqrt{\frac{\kappa}{m}}, \tag{3}$$

and that therefore the time during one oscillation, not spend between $-\delta$ and δ, is

$$t_2 < \frac{2L}{\delta}\sqrt{\frac{m}{\kappa}}. \tag{4}$$

(e) Argue that for sufficiently small values of ϵ, a good estimate of the oscillation period is

$$\tau = 2\sqrt{\frac{m}{\kappa}} \log\left(\frac{2\kappa L^2}{\epsilon}\right). \tag{5}$$

Exercise 3: Perturbations to potential **7 points**

In this exercise we will consider a particle with mass m moving in the potential

$$U(x) = \tfrac{1}{2}m\omega^2x^2 + \delta U(x), \tag{1}$$

where the δU will be considered a small perturbation.

(a) Let us first consider the unperturbed case where $\delta U(x) = 0$. In this case the problem is a standard harmonic oscillator.
A solution to the equations of motion for that case is

$$x_0(t) = A_0 \sin(\omega t)\,. \tag{2}$$

Calculate the total energy of the particle in terms of A_0, and use it to find an expression for A_0 as a function of the energy E.

(b) We have seen in class and in the previous exercises, that the time it takes to move between points x_1 and x_2, is

$$t = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dy}{\sqrt{E - U(y)}}\,. \tag{3}$$

Show that an equivalent expression is

$$t = -\sqrt{2m}\, \frac{\partial}{\partial E} \int_{x_1}^{x_2} \sqrt{E - U(y)}\, dy\,, \tag{4}$$

and show that this expression is valid also when x_1 and x_2 are the two solutions of $E = U(x)$. Hint: consider also the contributions from the derivatives of $x_i(E)$.

From eq. (4), one can derive that a perturbation in the potential $U \to U + \delta U$ implies a change in the period of oscillation $T \to T + \delta T$, with

$$\delta T \approx -\sqrt{2m}\, \frac{\partial}{\partial E} \int_{-A_0(E)}^{A_0(E)} \frac{\delta U(x)\, dx}{\sqrt{E - U_0(x)}} \tag{5}$$

where A_0 is the upper turning point of the oscillation, in the unperturbed case, cf. previous points.
Let us in the following focus on the cases where δU has the form of a power law:

$$\delta U(x) = \tfrac{1}{n} m \alpha x^n, \qquad (n \in \mathbb{N}) \tag{6}$$

where the perturbation parameter α is considered smaller than all other scales in the problem.

(c) What is the change to the orbital period δT for $n = 3$ and other odd values?

(d) Derive an expression for δT in terms of E, ω, m, and α, for the case of $n = 4$.

(e) What is the qualitative difference between the case of $n = 2$, and other even cases?