Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. H. Frellesvig, Dr. R. Rietkerk

Exercise Sheet 4

Issue: 11.05.18 – Submission: 18.05.18 at 09:30 – Discussion: 22.05.18

Exercise 1: Galilean transformations — 8 points

In the lectures, we have derived a general expression for the conserved quantity, that follows from the action being invariant under the infinitesimal transformation

$$q_i \to q_i' = q_i + \epsilon \Psi_i(q,t)\,, \qquad t \to t' = t + \epsilon X(q,t)\,, \tag{1}$$

with $q = \{q_i\}$. Let us now instead consider the case for which the action changes as

$$S = \int_{t_1}^{t_2} L(q, \partial q/\partial t, t) dt = \int_{t_1'}^{t_2'} \left(L(q', \partial q'/\partial t', t') + \epsilon \frac{df(q',t')}{dt'} \right) dt'\,, \tag{2}$$

where f is a function to be determined. Adding a total time-derivative to the Lagrangian leaves the physics unaltered, as was seen during the first weeks of the lectures.

(a) Show that for that case, the conserved quantity is

$$I = LX + \sum_i \frac{\partial L}{\partial \dot{q}_i}(\Psi_i - \dot{q}_i X) + f(q,t)\,. \tag{3}$$

Hint: follow the same steps as in the lecture.

(b) Let us now look at the case of a (infinitesimal) Galilei transformation $q_i \to q_i + \epsilon w_i t$. What is the physical interpretation of this transformation and of the w_i?

(c) Derive an expression for the conserved quantity I_g for a free particle of mass m, that follows from the action being invariant under Galilei transformations. Hint: identify Ψ_i, X, and f.

(d) Show that this implies the conservation of

$$\chi_i = m q_i - p_i t\,, \tag{4}$$

for $i \in \{1,2,3\}$. ($p_i = m\dot{q}_i$ denotes the momentum).

(e) A free particle conserves energy (one equation), momentum (three equations), and angular momentum (three equations). Additionally the three quantities of eq. (4) are conserved. That is 10 integrals of motion in total that characterizes the physics. But the movement of the particle is completely specified by six boundary conditions (e.g. the position and velocity at $t = 0$). Why does the existence of 10 integrals of motion not over-determine the system?

Exercise 2: Conservation of the Runge-Lenz vector **12 points**

In this exercise we are going to derive the conservation under certain conditions, of a quantity known as the Runge-Lenz vector $\vec{A}$. We will start from the expression for a conserved quantity derived in the previous exercise,

$$I = LX + \sum_i \frac{\partial L}{\partial \dot{q}_i}(\Psi_i - \dot{q}_i X) + f\,, \tag{1}$$

valid whenever the physics is invariant under

$$\begin{aligned} q_i \to q_i' = q_i + \epsilon \Psi_i(q,t)\,, \qquad t \to t' = t + \epsilon X(q,t)\,, \\ L(q,\dot{q},t) \to L(q',\dot{q}',t') + \epsilon \frac{df(q',t')}{dt'}\,. \end{aligned} \tag{2}$$

(a) Let us consider a transformation with $\Psi_i = \beta_i(t)$ and $X = 0$. We will for now leave the function f free, to be determined later. What is the conserved quantity I_A corresponding to invariance under that transformation?

(b) I_A being conserved implies that $\frac{dI_A}{dt} = 0$. For a particle of mass m in a potential $U(q)$, show that this quantity becomes

$$\frac{dI_A}{dt} = \sum_i \left(\left(m\dot{\beta}_i + \frac{\partial f}{\partial q_i} \right) \dot{q}_i - \beta_i \frac{\partial U}{\partial q_i} \right) + \frac{\partial f}{\partial t}\,. \tag{3}$$

(c) We want to use eq. (3) to find an expression for f. Having f being independent of $\dot{q}$ as we assumed, implies that the coefficient of each $\dot{q}_i$ in eq. (3) has to equal zero individually. Show that a solution for f is

$$f(q,t) = -m\dot{\beta}_i q_i\,. \tag{4}$$

(d) Find the expression for I_A with the expression for f inserted.

(e) Let us now assume a spherically symmetric potential, such that $U(q) = U(r)$ with $r = \sqrt{q_i^2}$. Show that the β_i have to fulfill the differential equation

$$\ddot{\beta}_i + \frac{1}{mr}\frac{\partial U}{\partial r}\beta_i = 0\,. \tag{5}$$

We will now consider the particle moving along a specific path $q_i = x_i(t)$ that is a solution to the equations of motion for the system.

(f) Let us now consider the Kepler problem, i.e. the potential $U(r) = \frac{-k}{r}$. Show that a solution to eq. (5) is $\beta_i(t) = \kappa_i \sum_j x_j(t)\dot{x}_j(t)$ where the κ_i are free constants.
Hint: identify and use the equations of motion for this potential.

(g) Show that for that choice of β_i, I_A becomes

$$I_A = \sum_i \kappa_i \left(m\dot{x}_i x_j \dot{x}_j + (k/r - m\dot{x}^2)x_i \right)\,. \tag{6}$$

(h) The Runge-Lenz vector is defined as

$$\vec{A} = \vec{p} \times \vec{M} - mk\frac{\vec{r}}{r}\,, \tag{7}$$

where $\vec{p}$ is the momentum vector, and $\vec{M} = \vec{r} \times \vec{p}$ is the angular momentum vector. The Runge-Lenz vector is conserved in the Kepler problem. Prove this, by showing that the conservation of the I_A of eq. (6) is equivalent to the conservation of the Runge-Lenz vector.

The Runge-Lenz vector being conserved in the Kepler problem, makes it useful in problems of orbital mechanics. In a future problem-set we are going to see the Runge-Lenz vector applied to such a problem.