Classical Theoretical Physics II

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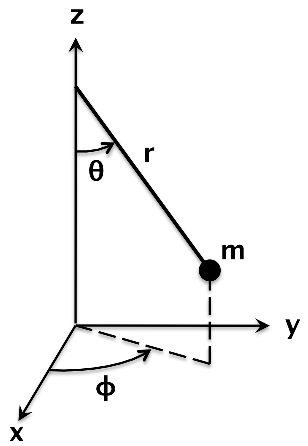
Exercise Sheet 3

Issue: 04.05.18 – Submission: 11.05.18 before 09:30 – Discussion: 15.05.18

Exercise 1: Spherical Pendulum — 5 points

A spherical pendulum consists of a massless rod of length r, which is attached to a fixed point at one end and has a mass m on the other end. The pendulum is free to move, under the influence of gravity, in two directions around the fixed point. The position of the mass is thus described by spherical coordinates (θ, ϕ).



(a) Construct the Lagrangian for this system.

(b) The Lagrangian is independent of time t. What is the corresponding conserved quantity? Derive its expression, starting from the general formula for the conserved Noether charge,

$$I = \sum_i \frac{\partial L}{\partial \dot{q}_i} \left(\Psi_i - X\dot{q}_i\right) + L\,X \ , \tag{1}$$

corresponding to a generic symmetry transformation $t \to t + \epsilon X(\{q_j\}, t)$, $q_i \to q_i + \epsilon \Psi_i(\{q_j\}, t)$.

(c) The Lagrangian is also independent of the coordinate ϕ. What is the related symmetry of the action? Derive the conserved quantity from eq. (1). What is the physical interpretation of this conserved quantity?

(d) Derive the Euler-Lagrange equations. Which equation leads to the conserved quantity of the preceding question?

(e) Suppose that $\theta = \theta_0$ is constant. Show that the pendulum revolves around the vertical axis with constant angular velocity

$$\dot{\phi}_0 = \sqrt{\frac{g}{r\cos\theta_0}} \ . \tag{2}$$

Exercise 2: Conserved quantities in various potentials **10 points**

Consider a particle of mass m with position $\vec{r} = (r_1(t), r_2(t), r_3(t))$, whose motion is described by the Lagrangian $L = \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r}, t)$. We will consider various types of potentials $U(\vec{r}, t)$, in order to practice finding transformations that leave the action invariant and to compute the associated conserved quantities.

(a) Suppose that the potential has the form $U(\vec{r}, t) = U(\vec{r} - \vec{v}_0 t)$, where $\vec{v}_0$ is a constant vector. What infinitesimal transformation, of the form

$$\begin{aligned} r_i &\to r_i + \epsilon\Psi_i \quad (i = 1, 2, 3) \ , \\ t &\to t + \epsilon X \ , \end{aligned} \tag{1}$$

leaves the Lagrangian (and thus the action) invariant? Show that the associated conserved quantity is

$$E(t) - \vec{p} \cdot \vec{v}_0 = \text{const} \ , \tag{2}$$

where the energy $E(t) = T + U = \frac{1}{2}m\dot{r}_i^2 + U$ is actually *not* constant.

(b) Suppose instead that the potential has the form $U(\vec{r}, t) = -\vec{F} \cdot \vec{r}$, where $\vec{F}$ is a constant force. Obviously, the Lagrangian does not depend explicitly on time, so $t \to t + \epsilon$ is a symmetry of the action and energy is conserved. Find two additional symmetry transformations of the action, involving transformations of coordinates only, that lead to conservation of momentum in the plane perpendicular to $\vec{F}$ and to conservation of angular momentum around the direction of $\vec{F}$, respectively.

(c) Suppose that the potential is independent of time (so E is conserved) and that the potential satisfies $U(\lambda\vec{r}) = \lambda^{-2}U(\vec{r})$. Find an infinitesimal transformation of both r_i and t, as in eq. (1), that leaves the action (but not the Lagrangian) invariant. Show that the associated conserved quantity is

$$\vec{p} \cdot \vec{r} - 2Et = \text{const} \ . \tag{3}$$

(d) Consider the potential $U(\vec{r}) = -k/r^2$, with $k > 0$. Confirm that this is a special case of the previous question, so that eq. (3) holds true. Since this is a central potential, the angular momentum $\vec{M} = \vec{r} \times \vec{p}$ is also conserved. Show that the third conserved quantity, energy, can be expressed as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{k}{r^2} \ . \tag{4}$$

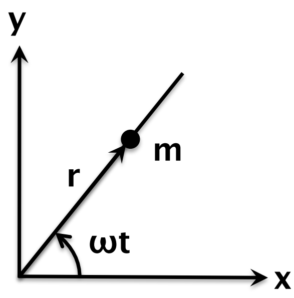
Finding all these conserved quantities is very useful for determining the trajectory in a relatively easy way, without solving differential equations. Indeed, combine eqs. (3) and (4) to derive the path in the radial direction,

$$r(t) = \sqrt{\frac{(2Et + \text{const})^2 + M^2 - 2mk}{2mE}} \ . \tag{5}$$

Hint: show that $\vec{p} \cdot \vec{r} = mr\dot{r}$.

Exercise 3: Bead on a rotating rod **5 points**

A bead of mass m is free to slide along a massless rod. The rod rotates in the (x, y)-plane with constant angular velocity ω.



(a) Construct the Lagrangian for this system.

(b) Compute the conserved quantity that is generated by the fact that the Lagrangian does not depend explicitly on time.

(c) Is the conserved quantity in the previous question equal to the energy of the bead m? Why yes, or why not?

(d) Find and sketch the path $r(t)$, subject to the boundary conditions $r(0) = r_0 > 0$ and $\dot{r}(0) = 0$.