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SuSe 2018

Classical Theoretical Physics II

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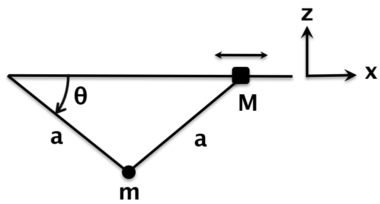
Exercise Sheet 2

Issue: 27.04.18 – Submission: 04.05.18 before 09:30 – Discussion: 08.05.18

Exercise 1: Sliding mass

3 points



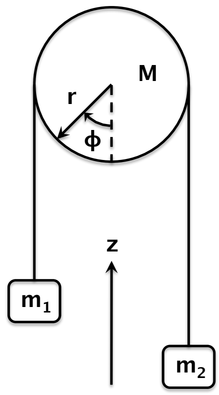
In this system, a bead with mass M is free to slide along a horizontal rod. A ball with mass m is attached with massless rods of length a. Gravity acts along the vertical direction.

(a) Construct the Lagrangian $L(\theta, \dot{\theta})$ for this system.

(b) Derive the Euler-Lagrange equation.

Exercise 2: Atwood's machine

7 points



Atwood's machine consists of two weights with masses m_1 and m_2, connected via a massless rope of length ℓ around a pulley with radius r and mass M. As one of the masses moves downward, the other mass is pulled upwards. The pulley rotates accordingly, so that the rope does not slip. The aim of this problem is to practice with the method of Lagrange multipliers.

(a) The kinetic energy of the rotating pulley is given by

$$T_{\text{pulley}} = \tfrac{1}{4} M r^2 \omega^2 \, , \tag{1}$$

where $\omega = \dot{\phi}$ is the angular velocity. (This will be derived later in the course.) Show that the no-slipping condition implies that $\omega = \dot{z}_1 / r$, where z_1 is the vertical position of mass m_1.

(b) Construct the Lagrangian $L(z_1, \dot{z}_1, z_2, \dot{z}_2)$, where z_2 is the vertical position of mass m_2. (The center of the pulley remains at a fixed position.)

(c) Give the constraint that the rope has a fixed length ℓ in the form $f(z_1, z_2) = 0$.

(d) Derive the Euler-Lagrange equations from $L_{\text{tot}} = L(z_1, \dot{z}_1, z_2, \dot{z}_2) + \lambda f(z_1, z_2)$.

(e) Eliminate the Lagrange multiplier λ from the Euler-Lagrange equations and impose the constraint $f(z_1, z_2) = 0$, to derive the acceleration of mass m_1,

$$\ddot{z}_1 = \frac{(m_2 - m_1)g}{m_1 + m_2 + M/2} \; . \tag{2}$$

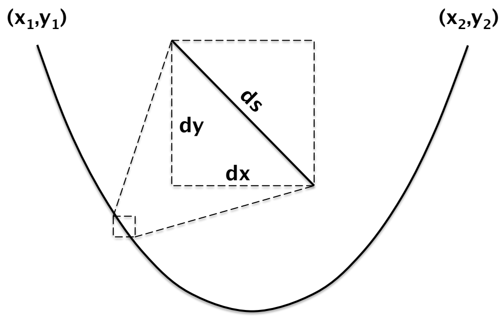
(f) The total force acting on mass m_2 is the vector sum of the tension in the rope (upwards) and the gravitational force (downwards). Compute the tension in the rope attached to mass m_2.

(g) Can you give the physical interpretation of the Lagrange multiplier λ?

Exercise 3: Catenary

10 points

The aim of this exercise is to determine the shape of a hanging rope (catenary) by means of variational calculus. Since this is not a dynamic system, the idea is simply to minimise the rope's potential energy.



(a) Denote the infinitesimal line element along the path of the rope (see figure) by ds. Show that the length ℓ of the rope is given by

$$\ell = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx \;, \quad y' = \frac{dy}{dx} \; . \tag{1}$$

(b) Let μ denote the constant mass density of the rope. Show that the potential energy of the rope is given by

$$U = \int \mu g\, y\, ds = \mu g \int_{x_1}^{x_2} y \sqrt{1 + y'^2}\, dx \; . \tag{2}$$

(c) The rope will take the shape $y(x)$ that minimises U with respect to small variations in $y(x)$, under the condition that the length of the rope is constant. This problem can be formulated, with Lagrange multiplier λ, as

$$\delta \int_{x_1}^{x_2} f(y, y')\, dx = 0 \;, \quad f(y, y') = (\mu g\, y - \lambda)\sqrt{1 + y'^2} \; . \tag{3}$$

Carry out the variation in eq. (3) for general function f to derive

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \; . \tag{4}$$

(d) Show that, since in this case $\partial f/\partial x = 0$, this equation is equivalent to

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \ . \tag{5}$$

Hint: start by writing out the total derivative df/dx.

(e) Integrate eq. (5) once with respect to x, calling the integration constant A. Derive from there the following differential equation for y,

$$\frac{\mu g\, y - \lambda}{\sqrt{1 + y'^2}} = A \ . \tag{6}$$

(f) Verify that this differential equation is solved by

$$y(x) = \frac{A}{\mu g}\cosh\left(\frac{\mu g}{A}(x - B)\right) + \frac{\lambda}{\mu g} \ , \tag{7}$$

where B is another integration constant.

(g) Insert the solution for $y(x)$ from eq. (7) into the integral in eq. (1), and calculate the length ℓ in terms of the unknown constants A, B and λ as well as the known parameters μ, g, x_1 and x_2.

(h) The three unknown constants A, B and λ can be fixed by three conditions: two equations to fix the endpoints of the rope, $y(x_1) = y_1$ and $y(x_2) = y_2$, and one equation to fix the length of the rope to a given value ℓ. Now suppose that the endpoints of the rope are attached at $x_1 = -1$ and $x_2 = +1$ at equal heights $y(x_1) = y(x_2) = 0$. Determine B and λ from the first two conditions. Derive from the third condition that A satisfies the equation

$$A \sinh\left(\frac{\mu g}{A}\right) - \frac{\mu g \ell}{2} = 0 \ . \tag{8}$$

Sketch the curve $y(x)$, once with $\ell = 3$ and once with $\ell = 100$.
Take $\mu = 1$ and $g = 10$. You may also use

$$z \sinh(10/z) - 15 = 0 \quad \Rightarrow \quad z \simeq 6.1647 \ , \tag{9}$$

$$z \sinh(10/z) - 500 = 0 \quad \Rightarrow \quad z \simeq 1.5449 \ . \tag{10}$$