Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. H. Frellesvig, Dr. R. Rietkerk

Exercise Sheet 13

Issue: 13.07.18 – Submission: 20.07.18 before 09:30

Exercise 1: Plate — 8 points

We borrow from the TTP-kitchen a thin circular plate of radius R, and with a homogeneous (areal) mass density such that the mass of the plate is m. We will position the plate in the x, y-plane, with its center in the origin.

(a) What is the (scalar) moment of inertia of the plate around the z-axis?

(b) What is the (scalar) moment of inertia of the plate around the x-axis? And around the y axis?

(c) Consider an axis parallel with the z-axis but touching the edge of the plate. Calculate the (scalar) moment of inertia of the plate around that axis using the parallel axis theorem.

(d) Repeat the previous question by performing the integral directly, and get agreement.

(e) Consider now drilling a hole (with radius q) with its center at $(r, 0, 0)$, such that $q < r$ and $q + r < R$. What are now the (scalar) moments of inertia of the plate around the x, y, and z axes?

(f) Consider now the plate (without the hole) rotating around the x-axis, with angular velocity ω. What is the kinetic energy of the plate?

(g) Consider now the plate (without the hole) rolling along a floor, with velocity v. What is the kinetic energy of the plate?

Exercise 2: Polygon — 4 points

Consider a homogeneous thin regular polygon with mass m, area A and N sides.

(a) Calculate the (scalar) moment of inertia I_N of the polygon with respect to the axis perpendicular to the polygon passing through its center.

(b) Show that the general result for the previous question reproduces the moments of inertia for the square and the circle:

$$I_{\text{square}} = \frac{mA}{6}\,, \qquad I_{\text{circle}} = \frac{mA}{2\pi}\,. \tag{1}$$

Hint: use the fact that $\lim_{N\to\infty} N\tan(\pi/N) = \pi$.

Exercise 3: Rocking Chair

8 points

After having worked all day with plates and polygons we take some rest in a rocking chair. It has a mass m and moment of inertia I_{cm} around its center of mass. The legs of the chair are wooden arcs with radius of curvature R. When the chair stands up straight, the center of mass is at a height $h < R$ straight above the point of contact with the floor. The aim of this problem is to determine the 'rocking frequency' of the chair.

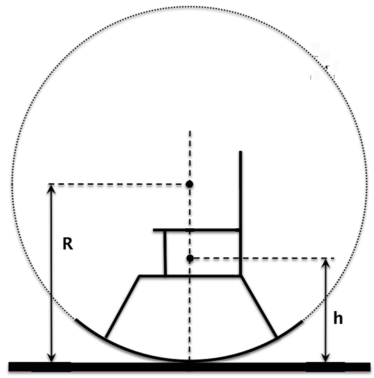

Figure 1: Rocking chair.

(a) Find the position of the center of mass $(x_{\text{cm}}(\theta), y_{\text{cm}}(\theta))$ as a function of the angle θ between the rocking chair and the vertical. Define the angle in such a way, that $\theta = 0$ when the chair stands up straight and that $\theta > 0$ when the chair leans backward, see fig. 2. Choose the origin such that $(x_{\text{cm}}(0), y_{\text{cm}}(0)) = (0, h)$.

(b) Determine the potential energy of the rocking chair (due to gravity) as a function of θ. Perform a Taylor expansion of the potential energy around the equilibrium point. Why are small oscillations of the rocking chair around the equilibrium point stable?

(c) Determine the kinetic energy of the rocking chair as a function of θ. Taylor expand it around the equilibrium point.

(d) Show that the frequency of small oscillations of the rocking chair is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{mg(R-h)}{I_{\text{cm}} + mh^2}}\,. \tag{1}$$

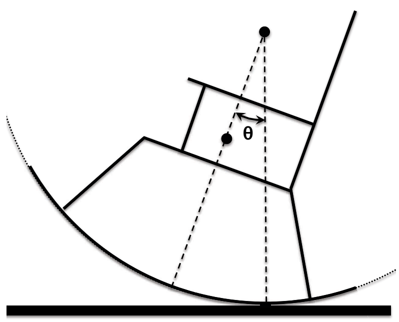

Figure 2: The backwards leaning rocking chair makes an angle θ with the vertical axis.

Solution of exercise 1: Plate

(a) 1 point *I around the z axis*
Let us start by finding the areal mass density

$$m = \int \rho dA = \int_0^R \int_0^{2\pi} \rho r d\theta dr = 2\pi\rho \int_0^R r dr = \pi\rho R^2 \Leftrightarrow$$
$$\rho = \frac{m}{\pi R^2} \ . \tag{2}$$

We may then find the moment of inertia around the z axis:

$$I_z = \int \rho r^2 dA = \frac{m}{\pi R^2} 2\pi \int_0^R r^3 dr = \frac{2m}{R^2}\frac{1}{4}R^4 = \frac{1}{2}mR^2 \ . \tag{3}$$

(b) 1 point *I around the x and y axis*
Around the x axis:

$$I_x = \int \rho y^2 dA = \frac{m}{\pi R^2}\int_{-R}^R dx \ 2\int_0^{\sqrt{R^2-z^2}} dy \ y^2 = \frac{2m}{\pi R^2}\int_{-R}^R dx \tfrac{1}{3}(\sqrt{R^2-x^2})^3$$
$$= \frac{2mR^2}{3\pi}\int_{-1}^1 (\sqrt{1-\xi^2})^3 d\xi = \frac{mR^2}{4} \ . \tag{4}$$

The problem is symmetric is x and y, so around the y axis the result is clearly the same, $I_y = \frac{mR^2}{4}$

(c) 1 point *I around new vertical axis*
The parallel axis theorem says

$$I_v = md^2 + I_{\text{cm}} = mR^2 + \frac{1}{2}mR^2 = \frac{3}{2}mR^2 \tag{5}$$

(d) 1 point *I around new vertical axis by direct calculation*
We can parametrise the circle by k and ϕ where k is the distance to the axis. "Thales' theorem" tells us that the upper limit of the k integration is $2R\sin\phi$. Therefore

$$I_v = \int \rho k^2 dA = \frac{m}{\pi R^2}\int_{-\pi/2}^{\pi/2} d\phi \int_0^{2R\sin\phi} dk \, k^3 = \frac{4mR^2}{\pi}\int_{-\pi/2}^{\pi/2} d\phi \sin^4\phi$$
$$= \frac{3mR^2}{2} \tag{6}$$

(e) 2 points *Drilled hole*
The trick here is to consider the hole as another plate with negative mass density glued onto the original plate. That hole-plate has the mass

$$-\rho\pi q^2 = -m\frac{q^2}{R^2} \tag{7}$$

Around its own center of mass it has

$$I_{hz} = -m\frac{q^4}{2R^2} \qquad I_{hx} = I_{hy} = -m\frac{q^4}{4R^2} \tag{8}$$

and taking into account the displacement from the center of the original plate, it has around that point

$$I_{hz} = -m\frac{q^2 r^2}{R^2} - m\frac{q^4}{2R^2}\,, \qquad I_{hx} = -m\frac{q^4}{4R^2}\,, \qquad I_{hy} - m\frac{q^2 r^2}{R^2} - m\frac{q^4}{4R^2} \tag{9}$$

This means that the combined plate has moments of inertia that are the sum of the two:

$$\begin{aligned} I_z &= \frac{mR^2}{2}\left(1 - \frac{q^4}{R^4} - \frac{2q^2r^2}{R^4}\right) \\ I_x &= \frac{mR^2}{4}\left(1 - \frac{q^4}{R^4}\right) \\ I_y &= \frac{mR^2}{4}\left(1 - \frac{q^4}{R^2} - \frac{4q^2r^2}{R^4}\right) \end{aligned} \tag{10}$$

If the hole is considered small on may discard the q^4 terms.

(f) 1 point *Rotating around x*

$$E = \tfrac{1}{2} I_x \omega^2 = \frac{mR^2\omega^2}{8} \tag{11}$$

(g) 1 point *Rolling*

When something rolls, the point that touches the floor is stationary, so $v = -\omega R$. The kinetic energy is given by

$$T = T_{\text{rot}} + T_{\text{trans}} = \frac{1}{2}mv^2 + \frac{1}{2}I_z\omega^2 = \frac{1}{2}mv^2 + \frac{1}{4}mR^2(-\tfrac{v}{R})^2 = \frac{3}{4}mv^2 \tag{12}$$

Solution of exercise 2: Polygon

(a) 2 points *Moment of inertia polygon*

Use the symmetry of the problem and split up the polygon into N triangles, each triangle having two corners at the endpoints of a given side of the polygon and the third endpoint at the center of the polygon. By symmetry, the moment of inertia of the polygon is equal to N times the moments of inertia of each of these triangles: $I_N = N I_{\text{tri}}$.

A given triangle has area A/N and opening angle (at the center) $2\pi/N$. Its moment of inertia is

$$I_{\text{tri}} = \int dm\; r^2 = \mu \int_0^b dy \int_{-y\tan(\pi/N)}^{y\tan(\pi/N)} dx\; (x^2 + y^2) \tag{13}$$

$$= 2\mu \int_0^b dy \int_0^{y\tan(\pi/N)} dx\; (x^2 + y^2)\,, \tag{14}$$

where μ is the constant mass density $\mu = m/A$ and b is the shortest distance between the center and the edge of the polygon, which satisfies $b^2 = (A/N)(1/\tan(\pi/N))$. Performing the integral is easy and gives

$$\begin{aligned} I_{\text{tri}} &= 2(m/A) \int_0^b dy \left(\frac{1}{3} y^3 \tan^3(\pi/N) + y^3 \tan(\pi/N) \right) \\ &= 2(m/A) \tan(\pi/N) \left(\frac{1}{3} \tan^2(\pi/N) + 1 \right) \frac{1}{4} b^4 \\ &= \frac{mA}{2N^2} \left[\frac{\tan(\pi/N)}{3} + \frac{1}{\tan(\pi/N)} \right] . \end{aligned} \tag{15}$$

Multiplying by N, we finally get

$$I_N = \frac{mA}{2} \left[\frac{\tan(\pi/N)}{3N} + \frac{1}{N \tan(\pi/N)} \right] . \tag{16}$$

(b) 2 points *Reproduce square and circle*
Setting $N = 4$ and using $\tan(\pi/4) = 1$, we find

$$I_{\text{square}} = I_4 = \frac{mA}{2} \left[\frac{1}{12} + \frac{1}{4} \right] = \frac{mA}{6} , \tag{17}$$

which is the moment of inertia of the square around the axes perpendicular to the square, passing through its center.
In the limit $N \to \infty$ the polygon becomes a perfect circle. In this limit, the term $\frac{\tan(\pi/N)}{3N}$ tends to zero, while $\frac{1}{N \tan(\pi/N)}$ tends to $1/\pi$. Therefore

$$I_{\text{circle}} = \lim_{N \to \infty} I_N = \frac{mA}{2} \left[0 + \frac{1}{\pi} \right] = \frac{mA}{2\pi} . \tag{18}$$

Solution of exercise 3: RockingChair

(a) 2 points *Position center of mass*

$$\begin{aligned} x_{\text{cm}}(\theta) &= R\theta - (R-h)\sin\theta , \\ y_{\text{cm}}(\theta) &= R - (R-h)\cos\theta , \end{aligned} \tag{19}$$

(b) 2 points *Potential energy*

$$U = mgy_{\text{cm}} = mg[R - (R-h)\cos\theta] \approx mgh + \frac{1}{2}(R-h)\theta^2 \tag{20}$$

Since we assumed that $h < R$, the potential energy is a parabola with a minimum at the equilibrium point $\theta = 0$. Since the potential has a minimum there, the motion of small oscillations around this equilibrium point is stable.

(c) 2 points *Kinetic energy*
Note that the kinetic energy has two contributions: from translational and rotational motion!

$$\begin{aligned}
T &= \frac{1}{2}m(\dot{x}_{\rm cm}^2 + \dot{y}_{\rm cm}^2)^2 + \frac{1}{2}I\dot{\theta}^2 \\
&= \frac{1}{2}m\dot{\theta}^2\left[\left(\frac{x_{\rm cm}}{d\theta}\right)^2 + \left(\frac{y_{\rm cm}}{d\theta}\right)^2\right] + \frac{1}{2}I\dot{\theta}^2 \\
&= \frac{1}{2}m\dot{\theta}^2\Big[R^2 + (R-h)^2 - 2R(R-h)\cos\theta\Big] + \frac{1}{2}I\dot{\theta}^2 \\
&\approx \frac{1}{2}mh^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2
\end{aligned} \tag{21}$$

Note that we drop the term θ^2 in the expansion of the cosine, because the prefactor $\dot{\theta}^2$ is already small in the small angle approximation.

(d) 2 points *Determine frequency*
Having the Lagrangian $L = T - U$, we compute (of course!) the Euler-Lagrange equation

$$(I + mh^2)\ddot{\theta} + mg(R-h)\theta = 0\ . \tag{22}$$

This equation may be written in the form $\ddot{\theta} + \omega^2\theta = 0$, which describes a harmonic oscillator with frequency

$$f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{mg(R-h)}{I+mh^2}}\ . \tag{23}$$