# Classical Theoretical Physics II

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## Exercise Sheet 12

Issue: 06.07.18 – Submission: 13.07.18 before 09:30 – Discussion: 17.07.18

### Exercise 1: Anharmonic oscillator

$\boxed{\text{8 points}}$

In this exercise we will consider an anharmonic oscillator, which is constructed from the harmonic case by giving $T$ and $U$ a small extra position-dependent contribution, such that the Hamiltonian becomes

$$H = \frac{p^2}{2m}(1 + \epsilon\beta x) + \frac{1}{2}mx^2\omega^2(1 + \epsilon\alpha x) \ , \tag{1}$$

where $\epsilon$ is a small, dimensionless constant. We will now make a canonical transformation to a new pair of canonical variables $X$ and $P$. We will do this with the generating function

$$F = \Phi(x, P) - XP \ , \tag{2}$$

that is defined such that the variation of the action remains invariant, or correspondingly

$$p\dot{x} - H(x, p) = P\dot{X} - K(X, P) + \frac{dF}{dt} \ , \tag{3}$$

where $K$ is the Hamiltonian for the new coordinates.

(a) Derive the transformation equations

$$p = \frac{\partial \Phi}{\partial x} \ , \qquad X = \frac{\partial \Phi}{\partial P} \ . \tag{4}$$

(b) What transformation is generated by $\Phi = xP$ ?

(c) We will now consider a generating function with

$$\Phi = xP + \epsilon a x^2 P + \epsilon b P^3 \ . \tag{5}$$

Find values of $a$ and $b$ such that $K(X, P)$ is the Hamiltonian of a harmonic oscillator up to terms of order $\epsilon^2$.

(d) Express $x(t)$ in terms of the known sinusoidal solutions of the harmonic oscillator.

**Exercise 2: Areas in phase space** **12 points**

Let us consider, again, the harmonic oscillator given by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m x^2 \omega^2 \,. \tag{1}$$

(a) Verify that

$$x = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \theta_0)\,, \qquad p = \sqrt{2Em}\cos(\omega t + \theta_0)\,, \tag{2}$$

are solutions to the Hamiltonian equations of motion.

(b) Draw the path of an oscillation in $(x, p)$ phase space, and indicate on the drawing the dependence on $E$ and $t$.

(c) If $m$ and $\omega$ are known exactly, but the energy is only known to be between $E_0$ and $E_0 + \Delta E$ and the time to be between $t_0$ and $t_0 + \Delta t$, what is the area of $(x, p)$ phase space in which the particle may be found? ($\Delta E$ and $\Delta t$ are small.)
Hint: Consider how infinitesimal area elements transform under variable changes.

(d) We will now do a canonical transformation to new variables $X$ and $P$, defined by the generating function

$$F = \frac{m\omega}{2} x^2 \cot(X)\,, \tag{3}$$

(Here, cot is the co-tangent $\cot(z) = \cos(z)/\sin(z)$.)
Find expression for $x$ and $p$ in terms of $X$ and $P$.

(e) Find expressions for $X$ and $P$.

(f) Draw the path of an oscillation in $(X, P)$ phase space, and determine the area in that phase space in which the particle may be found if it has the $\Delta E$ and $\Delta t$ uncertainties discussed above.