Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. H. Frellesvig, Dr. R. Rietkerk

Exercise Sheet 11

Issue: 29.06.18 – Submission: 06.07.18 before 09:30 – Discussion: 10.07.18

Exercise 1: Hamiltonians for physical problems — 5 points

Find the canonical momenta, the Hamiltonian and Hamilton's equations of motion for each of the following cases.

(a) A free particle with mass m in one dimension.

(b) A harmonic oscillator with mass m and angular frequency ω in one dimension.

(c) A particle with mass m in the one-dimensional potential $U(x) = \alpha x^n$.

(d) A particle with mass m in a three-dimensional potential $U(r) = -k/r$. Use polar coordinates r, θ, ϕ.

(e) Two particles with masses m and M moving in a two-dimensional plane and interacting gravitationally. Use Cartesian coordinates x_i, y_i for each particle. (What could have been a better set of coordinates for this problem?)

Exercise 2: Poisson brackets — 6 points

Consider a particle with mass m, three-dimensional coordinate $\vec{r}$ and momentum $\vec{p}$. Its angular momentum is given by $\vec{M} = \vec{r} \times \vec{p}$. The components of these vectors are denoted by r_i, p_i and M_i, and their lengths by r, p and M, respectively.

(a) Calculate the following Poisson brackets

$$\{M_i, r_j\} \ , \ \{M_i, p_j\} \ , \ \{M_i, M_j\} \ \text{ and } \ \{M_i, M^2\} \ . \tag{1}$$

(b) The Runge-Lenz vector is $\vec{A} = \vec{p} \times \vec{M} - mk\hat{r}$, with $\hat{r} = \vec{r}/r$. Prove that

$$\{M_i, A_j\} = -\epsilon_{ijk} A_k \ . \tag{2}$$

(c) Prove that the Runge-Lenz vector is conserved in the Kepler problem, by showing that $\{H, A_i\} = 0$ for the Hamiltonian $H = \frac{p^2}{2m} - \frac{k}{r}$.

Exercise 3: Particle in magnetic field

9 points

The motion of a particle with mass m and charge e is described by the Lagrangian

$$L = \tfrac{1}{2}m\dot{\vec{r}}^2 - \left(e\phi(\vec{r},t) - \frac{e}{c}\vec{A}(\vec{r},t)\cdot\dot{\vec{r}}\right) , \tag{1}$$

where ϕ and $\vec{A}$ are electromagnetic potentials.

(a) Determine the canonical momentum $\vec{p}$, construct the Hamiltonian H and show that Hamilton's equations are given by

$$\dot{r}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right) , \quad \dot{p}_i = \frac{e}{mc}\left(p_j - \frac{e}{c}A_j\right)\frac{\partial A_j}{\partial r_i} - e\frac{\partial \phi}{\partial r_i} . \tag{2}$$

(b) Derive from Hamilton's equations the Lorentz force law. Hint: use index notation and the following expressions for the electric and magnetic fields in terms of the electromagnetic potentials

$$E_i = -\frac{\partial \phi}{\partial r_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} , \quad B_i = \epsilon_{ijk}\frac{\partial A_k}{\partial r_j} . \tag{3}$$

(c) In the presence of a magnetic field we have that $\vec{p} \neq m\vec{v}$, where $\vec{v} = \dot{\vec{x}}$. This is evident from the first of Hamilton's equations in eq. (2). Their Poisson brackets also reflect this fact: while $\{p_i, p_j\} = 0$, show that

$$\{v_i, v_j\} = -\frac{e}{m^2 c}\epsilon_{ijk}B_k . \tag{4}$$

(d) Consider the specific case of a particle restricted to the x, y-plane, with potentials $\phi(\vec{r},t) = 0$ and $\vec{A}(\vec{r},t) = (-By, 0, 0)^T$. What are the $\vec{E}$ and $\vec{B}$ fields? Give Hamilton's equations for this case. Show that they imply

$$m\dot{y} + \frac{eB}{c}x = K_1 , \quad m\dot{x} - \frac{eB}{c}y = K_2 , \tag{5}$$

with unknown constants $K_{1,2}$.

(e) Show that the general solution to eq. (5) is given by

$$\begin{aligned} x(t) &= \frac{K_1}{m\omega} + R\sin(\omega t + \phi) , \\ y(t) &= -\frac{K_2}{m\omega} + R\cos(\omega t + \phi) , \end{aligned} \tag{6}$$

where $\omega = \frac{eB}{mc}$ and R and ϕ are unknown integration constants.

(f) Impose the boundary conditions $x(0) = x_0$, $y(0) = y_0$, $\dot{x}(0) = v_0$ and $\dot{y}(0) = 0$, and sketch the motion of the particle.