Classical Theoretical Physics II

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Exercise Sheet 10

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Exercise 1: Perturbations from orbit — 9 points

In this exercise we will investigate the consequence of perturbations away from a circular orbit under the influence of a central force.

(a) The particle (with mass m) moves in the plane described by polar coordinates r, θ, under the influence of a central force caused by the potential $U(r)$. Show that the Lagrangian is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$.

(b) Let us now look at the case where U is a power law $U(r) = -ar^{1-n}$. Draw the *effective* potential for various values of n.

(c) If the particle moves in a circular orbit with constant radius R and constant angular frequency Ω, what is the relation between R, Ω, n, m, and a ?

(d) Let us now consider a small perturbation away from the stable orbit, such that $r = R + \epsilon\rho$ and $\theta = \Omega t + \epsilon\phi$, where ϵ is a small parameter. Show that in this case, the Lagrangian can be written as

$$\begin{aligned} L = &\frac{1}{2}a(1+n)R^{1-n} + \epsilon m R^2 \Omega \dot{\phi} \\ &+ \frac{\epsilon^2}{2}\Big(aR^{-n-1}(n^2-1)\rho^2 + m(R^2\dot{\phi}^2 + \dot{\rho}^2 + 4\Omega R\dot{\phi}\rho)\Big) + \mathcal{O}(\epsilon^3)\,. \end{aligned} \tag{1}$$

(e) Derive the equations of motion for the particle in the new coordinates.

(f) Assume an oscillating solution to the EOM for ρ and ϕ of the form $\exp(i\omega t)$, such that $\dot{\rho} = i\omega\rho$, $\ddot{\rho} = -\omega^2\rho$, $\dot{\phi} = i\omega\phi$, $\ddot{\phi} = -\omega^2\phi$. Show that with this assumption, the EOMs can be written as $A_{11}\rho + A_{12}\phi = 0$ and $A_{21}\rho + A_{22}\phi = 0$ with

$$\begin{aligned} A_{11} &= a(1-n^2)R^{-1-n} - m\omega^2, & A_{12} &= -2i\omega m R\Omega\,, \\ A_{21} &= 2i\omega m R\Omega\,, & A_{22} &= -\omega^2 m R^2. \end{aligned} \tag{2}$$

(g) Show that the criterion for the existence of a non-zero solution to the system of EOMs (using the oscillating ansatz), is $A_{11}A_{22} = A_{12}A_{21}$.

(h) Show that this criterion implies $\omega = 0$ or $\omega = \pm\Omega\sqrt{3-n}$.

(i) The oscillating ansatz used, were on the form $\exp(i\omega t)$, so in general complex. Given the allowed values for ω, what are a set of real solutions?

(j) What is the physical interpretation of these solutions? How do we interpret the imaginary ω for $n > 3$?

Exercise 2: A more true pendulum

6 points

In this exercise, we will consider a pendulum of length R and mass m, as seen of fig. 1.

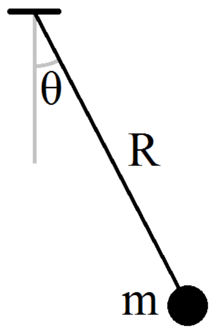

Figure 1

(a) Write down the Lagrangian for the pendulum in terms of the angle θ, and find the equation of motion.

(b) A common approximation to the equation of motion for the pendulum, is $\ddot{\theta} + \omega_0^2\theta = 0$ with $\omega_0^2 = g/R$. This approximation is valid whenever $\theta \ll 1$. In this exercise we will, however, discuss the next term in the expansion. Show that the EOM for small angles, can be approximated as

$$\ddot{\theta} + \omega_0^2\theta = \alpha\theta^3 \tag{1}$$

What is the expression for α?

(c) We will make an ansatz for the solution to that EOM, of the form

$$\theta(t) = A_0(\cos(\omega t) + \phi(t)) \qquad \text{with} \qquad \omega = \omega_0 + \omega_1 \tag{2}$$

under the assumptions $|\phi| \ll 1$ and $\omega_1 \ll \omega_0$.
Show that the EOM for ϕ can be approximated as

$$\ddot{\phi} + \omega_0^2\phi = (A_0^2\alpha/4)\cos(3\omega t) + (2\omega_0\omega_1 + 3A_0^2\alpha/4)\cos(\omega t) \tag{3}$$

Hint 1: The size hierarchy is such that A_0^2, ϕ, and ω_1 can be considered about equally small.
Hint 2: Use the relation between $\cos^3(x)$ and $\cos(3x)$.

(d) The EOM for ϕ implies that, physically, it has to be the case that

$$\omega_1 = \frac{-3A_0^2\alpha}{8\omega_0}. \tag{4}$$

Why is that?

(e) Find a solution for ϕ.

(f) Plot $\theta(t)/A_0$ for the harmonic solution $\theta(t) = A_0\cos(\omega_0 t)$, and for the anharmonic solution given by eq. (2), in the same graph. Do this for a few different values of A_0. Does this agree with what you would expect for the small angle approximation?

Exercise 3: Anharmonic kinetic term

5 points

A certain oscillator has a kinetic term with a small position-dependent contribution, such that

$$L = \tfrac{1}{2}m(1+\gamma x)\dot{x}^2 - \tfrac{1}{2}m\omega_0^2x^2 \tag{1}$$

We make a solution ansatz of the form

$$x(t) = A\cos(\omega t) + x_1(t) \tag{2}$$

where $\omega = \omega_0 + \omega_1$, and where $x_1(t)$ and ω_1 may be considered small, and of comparable size to γ.

(a) Using the methods of anharmonic oscillations, find expressions for ω_1 and $x_1(t)$ in terms of ω_0, γ, and A.