Classical Theoretical Physics II

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Exercise Sheet 1

Issue: 20.04.18 – Submission: 27.04.18 before 09:30 – Discussion: 30.04.18

Exercise 1: Falling object — 4 points

A mass m falls vertically in the earth's gravitational field. We describe the path with the ansatz $z(t) = c_0 + c_1 t + c_2 t^2$, where c_0, c_1 and c_2 are constants to be determined.

(a) Give the Lagrangian that describes this situation.

(b) Determine the constants c_0 and c_1 from the conditions $z(0) = h$ and $z(T) = 0$.

(c) Compute the action,

$$S = \int_0^T \mathrm{d}t \; L(z(t), \dot{z}(t)) \;, \tag{1}$$

as a function of m, g, h, T, c_2.

(d) Use the minimal action principle to derive that $c_2 = -g/2$. Is this the result you expected?

Exercise 2: Lagrangian with velocity-dependent potential — 4 points

Consider the Lagrangian for a particle with mass m in a velocity-dependent potential U,

$$L = \tfrac{1}{2} m \dot{\vec{r}}^{\,2} - U(\vec{r}, \dot{\vec{r}}, t) \;. \tag{1}$$

The velocity-dependence of the potential is parametrised by

$$U(\vec{r}, \dot{\vec{r}}, t) = e\,\phi(\vec{r}, t) - \frac{e}{c} \vec{A}(\vec{r}, t) \cdot \dot{\vec{r}} \;, \tag{2}$$

where e and c are constants of Nature, $\phi(\vec{r}, t)$ is an electric potential and $\vec{A}(\vec{r}, t)$ is a vector potential.

(a) Write down the Euler-Lagrange equations.

(b) Apply the following vector identity, derived in the exercise sheet 0,

$$\vec{\nabla}\,(\vec{x} \cdot \vec{y}\,) = (\vec{x} \cdot \vec{\nabla}\,)\,\vec{y} + (\vec{y} \cdot \vec{\nabla}\,)\,\vec{x} + \vec{y} \times (\vec{\nabla} \times \vec{x}\,) + \vec{x} \times (\vec{\nabla} \times \vec{y}\,) \;, \tag{3}$$

to rewrite the term $\frac{\partial}{\partial \vec{r}}\,(\vec{A} \cdot \dot{\vec{r}}\,) \equiv \vec{\nabla}\,(\vec{A} \cdot \dot{\vec{r}}\,)$ in the Euler-Lagrange equations.

(c) Show that the Euler-Lagrange equations take the form

$$m\ddot{\vec{r}} = -\frac{e}{c}\frac{\partial \vec{A}}{\partial t} - e\vec{\nabla}\phi + \frac{e}{c}\dot{\vec{r}} \times (\vec{\nabla} \times \vec{A}) \ . \qquad (4)$$

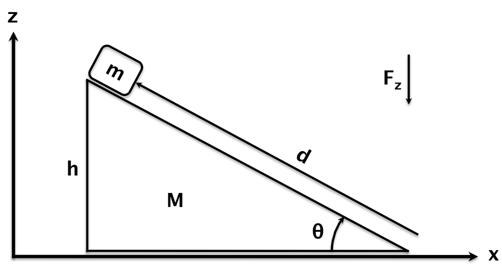
(d) In Maxwell's theory of electrodynamics, the electric and magnetic fields are expressed in terms of an electric potential ϕ and a vector potential $\vec{A}$. We have that

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \ , \quad \vec{B} = \vec{\nabla} \times \vec{A} \ . \qquad (5)$$

Use these formulas to rewrite eq. (4) in terms of $\vec{E}$ and $\vec{B}$. Can you identify the force on the right-hand side of eq. (4)?

Exercise 3: Sliding down a movable ramp — 8 points

A box with mass m is placed on a smooth ramp of mass M. The ramp has a fixed angle θ and rests on a smooth horizontal surface. As the box starts to slide down the ramp under the influence of gravity, the ramp itself moves frictionless to the left. The goal of this exercise is to determine how long it takes before the box reaches the bottom of the ramp.



(a) Construct the Lagrangian in terms of Cartesian coordinates (x_M, y_M) and (x_m, y_m) for the positions of the lower right-hand corners of the two objects.

(b) Argue that there is a freedom to choose $y_M = 0$. The remaining three coordinates are not independent. Introduce the generalised coordinate d for the position of the box along the ramp (see figure) and eliminate x_m and y_m from the Lagrangian.

(c) Derive both Euler-Lagrange equations.

(d) Solve the Euler-Lagrange equations for the accelerations $\ddot{x}_M$ and $\ddot{d}$.

(e) What results do you expect for $\ddot{x}_M$ and $\ddot{d}$ in the limits $\theta \to 0$ and $\theta \to \pi/2$? Does the result in (d) indeed tend to those limits?

(f) What is $\ddot{d}$ in the limit $m \ll M$ (for $\theta \approx 30°$)?

(g) Suppose that the box is placed at rest at the top of the ramp, at height h. How long does it take the box to reach the bottom of the ramp?

Exercise 4: Equations of motion in different coordinates $\boxed{\textbf{4 points}}$

Consider a Lagrangian $L(q, \dot{q}, t)$ that describes a certain system. Alternatively, the same system could be described in terms of different coordinates $q' = q'(q, t)$ with a Lagrangian $L'(q', \dot{q}', t)$.

(a) Start from the Euler-Lagrange equations for q and L and show that the Euler-Lagrange equations for q' and L' have a "canonical" form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{q}'} = \frac{\partial L'}{\partial q'} \ . \tag{1}$$

Hint: the following identities are useful $\frac{\partial \dot{q}'}{\partial q} = \frac{d}{dt}\frac{\partial q'}{\partial q}$ and $\frac{\partial \dot{q}'}{\partial \dot{q}} = \frac{\partial q'}{\partial q}$.

(b) Consider a mechanical system that depends on N coordinates. Can you generalise the proof in question (a) accordingly, with transformations $q'_i = q'_i(q_1, q_2, \ldots, q_n, t)$ for $i = 1, 2, \ldots, N$?