SLAC-PUB-8321, TTP99-51
hep-ph/9912391

The three-loop relation between the $\overline{\rm MS}$ and the pole quark masses

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The analytic relation between the $\overline{\rm MS}$ and the pole quark masses is computed to $\mathcal{O}(\alpha_s^3)$ in QCD. Using this exact result, the accuracy of the large β_0 approximation is critically examined and the implications of the obtained relation for semileptonic B decays are discussed.

I. INTRODUCTION

Quark masses, being input parameters of the Standard Model Lagrangian, play a very important role in high energy physics phenomenology. It is believed that the quark masses are generated through a spontaneous symmetry breaking mechanism caused by a non-zero vacuum expectation value of a Higgs field. Nevertheless, the quark masses remain free parameters of the Lagrangian and have to be determined by comparing theoretical predictions with experimental data.

One has to keep in mind, however, that quarks have not been observed as free particles and the notion of a quark mass relies on a theoretical construction. Different definitions of quark masses exist referring to different schemes for the renormalization of the QCD Lagrangian of the strong interactions. Two prominent mass definitions are the $\overline{\rm MS}$ mass $\overline{m}$ and the pole mass M. The pole mass M is the renormalized quark mass in the on-shell (OS) renormalization scheme, while the $\overline{\rm MS}$ mass is the renormalized quark mass in the modified minimal subtraction scheme [1,2] which is intimately related to the use of dimensional regularization [3].

These two renormalization schemes are used in different physical situations. In the $\overline{\rm MS}$ scheme, the renormalized mass and coupling constant depend explicitly on the renormalization scale μ which is often chosen to be of the order of the characteristic scale Q of a process. The scale evolution of the $\overline{\rm MS}$ mass and the coupling constant can be done accurately by integrating the renormalization group equations, even for relatively low Q. However, the $\overline{\rm MS}$ mass is by construction sensitive to only the short distance (Euclidean) aspects of QCD. For processes in which the characteristic scale is large compared to the quark masses it is therefore advantageous to adopt the $\overline{\rm MS}$ scheme for the quark mass renormalization[1].

On the other hand, for processes where the characteristic scale is set by the mass of a quark in the initial or final state, the situation is different as long-distance aspects of QCD become important. Since in such cases the quarks are close to the mass shell, the on-shell scheme is a natural renormalization scheme and the pole quark mass emerges. Explicit perturbative calculations [5–7] have shown that the pole quark mass is an infrared finite, gauge invariant quantity[2] and for this reason the pole mass of a heavy quark has often been considered as a meaningful physical parameter with a corresponding numerical value.

It should be anticipated however that the pole mass of a quark is not a physical quantity in a truly non-perturbative sense since the confiment of quarks in QCD implies that there is no pole in the quark propagator beyond the perturbation theory. For this reason it is natural to expect that the pole quark mass might be sensitive to long distance effects responsible for keeping quarks and gluons confined. This expectation was confirmed recently when it became clear that the infrared sensitivity of the pole mass reveals itself through contributions in the perturbative series for the pole mass that grow factorially at higher orders corresponding to a singularity close to the origin in the Borel plane [9,10]. This renormalon singularity is important for phenomenology since it implies that, indeed, the pole quark mass can not be determined with an accuracy better than $\delta M \approx \Lambda_{\rm QCD}$ [9,10].

Taken literary, the renormalon analysis applies to asymptotically high orders in perturbation theory. However, since the location of the leading renormalon pole in the Borel plane can be correctly obtained in the so-called large β_0 approximation and since in the lowest non-trivial order of perturbation theory the terms proportional to β_0 dominate, it was conjectured that the leading renormalon

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[1] This is demonstrated for instance in the extraction of the $\overline{\rm MS}$ b-quark mass, renormalized at the scale M_Z, with a relatively small uncertainty from three jet events at LEP [4].

[2] This result is proven to all orders in the coupling constant in Refs. [5,8].

dominates the growth of the perturbative coefficients already in low orders for the pole mass.

It is fair to say that this conjecture is an important ingredient in our understanding of theoretical uncertainties for various low scale physical processes such as semileptonic decays of B mesons, determination of the b quark mass from Υ mesons spectrum or the top quark threshold production cross section. In all these cases eliminating the pole quark mass from theoretical formulas is an important step in improving the convergence of the perturbative series. Nevertheless, it should be kept in mind that, up to date, this physical picture has been checked against only one non-trivial order of perturbation theory.

In spite of its infrared sensitivity, the pole mass remains to be important for processes where the on-shell mass definition has technical advantages. The pole mass can afterwards be eliminated in favor of a mass that is free from long distance ambiguities such as e.g. the $\overline{\mathrm{MS}}$ mass. It often happens that the $\overline{\mathrm{MS}}$ mass is not the best choice to parameterize the low scale processes and recently other short distance low scale masses have been proposed [11–13]. An attractive feature of these masses is that they can be accurately determined from the analysis of the low energy data [14–16]. However, for technical reasons, the relation of these masses to the $\overline{\mathrm{MS}}$ mass goes via the pole mass and therefore a determination of the $\overline{\mathrm{MS}}$ mass from the low scale short distance masses requires the knowledge of the conversion formula between $\overline{\mathrm{MS}}$ and the pole mass.

Hence, the relation between the $\overline{\mathrm{MS}}$ mass and the pole quark mass is of prime phenomenological importance. In the one-loop order this mass relation was obtained in [5]. The first analytical calculation of the two-loop contribution was performed in [6] and this result was confirmed in [7]. The three-loop coefficient was first estimated in Ref. [17] using the large β_0 approximation. Recently a more accurate result was obtained using asymptotic expansions and Pade improvements [18].

In this letter we present the analytical calculation of the three-loop relation between the $\overline{\mathrm{MS}}$ and the pole mass. Using this exact result, we critically examine the validity of the large β_0 approximation and discuss some implications of the obtained relation for semileptonic B decays. A more detailed description of our calculation will be published elsewhere.

II. THE THREE-LOOP RELATION BETWEEN THE $\overline{\mathrm{MS}}$ AND THE POLE MASSES

The quark masses in both, the $\overline{\mathrm{MS}}$ and the on-shell, renormalization schemes are renormalized multiplicatively and the connection between renormalized and bare quark masses is defined as

$$
\begin{aligned}
m_0 &= Z_m^{\overline{\mathrm{MS}}}\,\overline{m}, && (1)\\
m_0 &= Z_m^{\mathrm{OS}} M, && (2)
\end{aligned}
$$

where m_0 is the bare (unrenormalized) quark mass, and $Z_m^{\overline{\mathrm{MS}}}$ and Z_m^{OS} are the renormalization factors for the quark mass in, respectively, the $\overline{\mathrm{MS}}$ and the on-shell schemes. The relation between the pole quark mass and the $\overline{\mathrm{MS}}$ mass is then expressed as the ratio

$$\frac{\overline{m}}{M} = \frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}}. \qquad (3)$$

One sees that in order to calculate the relation between the $\overline{\mathrm{MS}}$ and the pole quark mass in the three-loop order one needs to calculate the mass renormalization factors in both the $\overline{\mathrm{MS}}$ and the on-shell schemes in the three-loop order. Fortunately, due to the very simple form of renormalization factors in the $\overline{\mathrm{MS}}$ scheme, the mass renormalization factor $Z_m^{\overline{\mathrm{MS}}}$ is presently known to a sufficiently high order in perturbation theory and can be taken from the literature [19]. One obtains:

$$Z_m^{\overline{\mathrm{MS}}} = 1 + \sum_{i=1}^{\infty} C_i \left(\frac{\alpha_s(\mu)}{\pi}\right)^i, \qquad (4)$$

with

$$
\begin{aligned}
C_1 &= -\frac{1}{\varepsilon},\\
C_2 &= \frac{1}{\varepsilon^2}\left(\frac{15}{8} - \frac{1}{12}N_f\right) + \frac{1}{\varepsilon}\left(-\frac{101}{48} + \frac{5}{72}N_f\right),\\
C_3 &= \frac{1}{\varepsilon^3}\left(-\frac{65}{16} + \frac{7}{18}N_f - \frac{1}{108}N_f^2\right)\\
&\quad + \frac{1}{\varepsilon^2}\left(\frac{2329}{288} - \frac{25}{36}N_f + \frac{5}{648}N_f^2\right)\\
&\quad + \frac{1}{\varepsilon}\left(-\frac{1249}{192} + \frac{5}{18}\zeta_3 N_f + \frac{277}{648}N_f + \frac{35}{3888}N_f^2\right), \qquad (5)
\end{aligned}
$$

where N_f denotes the number of different fermion flavors. The dimensional regularization parameter ε is defined through $\varepsilon = (4-D)/2$ with D being the space-time dimension.

In comparison to the $\overline{\mathrm{MS}}$-scheme, the calculation of renormalization factors in the on-shell scheme is much more involved and for this reason the renormalization factor Z_m^{OS} is known only in the two-loop approximation. In the present work we calculate Z_m^{OS} in the three-loop order of QCD, thereby obtaining the $\overline{\mathrm{MS}}$ pole mass relation in the three-loop order.

Let us explain how the on-shell renormalization factor Z_m^{OS} is computed. This renormalization constant follows from considering the one particle irreducible quark self-energy parameterized as

$$\hat{\Sigma}(p, M) = M\Sigma_1(p^2, M) + (\hat{p} - M)\Sigma_2(p^2, M). \qquad (6)$$

Here M is the pole-quark-mass and it is understood that the mass renormalization is performed in the on-shell scheme which means that on-shell mass counterterms proportional to $M\Sigma_1(M^2, M)$ are inserted in the diagrams of lower order in α. For the renormalization of the strong coupling constant we adopt the minimal subtraction scheme

$$\frac{\alpha_0}{\pi} = \frac{\alpha}{\pi} - \frac{\beta_0}{\varepsilon}\left(\frac{\alpha}{\pi}\right)^2 + \left(\frac{\beta_0^2}{\varepsilon^2} - \frac{\beta_1}{2\varepsilon}\right)\left(\frac{\alpha}{\pi}\right)^3 + O(\alpha^4), \quad (7)$$

where $\beta_0 = 11/4 - 1/6 N_f$ and $\beta_1 = 51/8 - 19/24 N_f$ are the first two coefficients of the QCD beta function. To make the connection with the formal on-shell mass renormalization factor $Z_m^{\rm OS}$ we note that the perturbative quark propagator can be written as

$$\hat{S}_F(p) = \frac{i}{\hat{p} - m_0 + \hat{\Sigma}(p, M)}. \quad (8)$$

Since the pole mass of the quark M corresponds to the position of the pole of the quark propagator one obtains

$$Z_m^{\rm OS} = 1 + \Sigma_1(p^2, M)\big|_{p^2=M^2} \quad (9)$$

which agrees with the form of the on-shell mass counterterms discussed above.

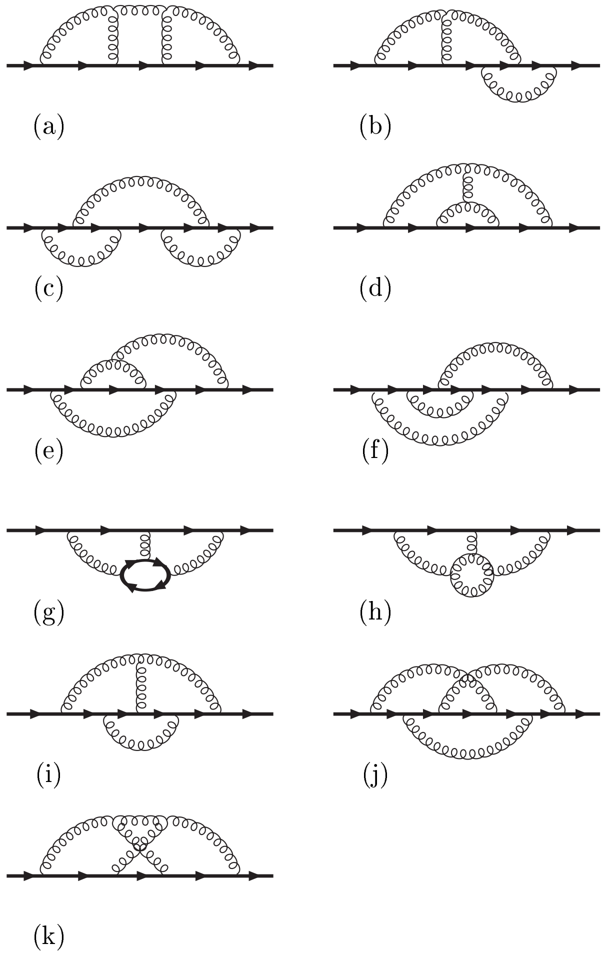

FIG. 1. Examples of three-loop quark propagator diagrams corresponding to eleven integration topologies.

Equation (9) provides the simplest formula for the calculation of $Z_m^{\rm OS}$. One computes $\Sigma_1(M^2, M)$ to the required order in perturbation theory taking into account the lower order difference between m_0 and M by calculating lower order diagrams with the appropriate mass counterterm insertions. Therefore, in order to obtain the three-loop on-shell renormalization factor $Z_m^{\rm OS}$ one has to compute on-shell propagator-type diagrams up to three loops.

The most efficient way to evaluate those multiloop integrals is to utilize integration-by-parts identities within dimensional regularization. There are eleven basic topologies that should be considered[3]. They are shown in Fig.1. For each of the topologies one writes down a system of recurrence relations based on integration-by-parts identities [3,22]. Solving this system, it is possible to show that any integral which belongs to the above topologies can be expressed through 18 master integrals. Fortunately, most of these integrals have been computed in the course of the analytical calculation of the electron anomalous magnetic moment [23] and can be taken from there. As compared to that reference, we need one additional master integral and we also need one of the master integrals of Ref. [23] to a higher order in the regularization parameter ε. For completeness, we present here the results for these two master integrals. Let us introduce $D_1 = k_1^2, D_2 = k_2^2, D_3 = k_3^2, D_4 = (k_1 - k_2)^2, D_5 = (k_3 - k_2)^2, D_6 = k_1^2 + 2pk_1, D_7 = k_2^2 + 2pk_2, D_8 = k_3^2 + 2pk_3$ with $p^2 = -1$. Then:

$$I_{18} = \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3}{D_1 D_3 D_4 D_5 D_6 D_7 D_8} = C(\varepsilon)\left(2\pi^2\zeta_3 - 5\zeta_5\right),$$

where $C(\varepsilon) = \left[\pi^{(2-\varepsilon)}\Gamma(1+\varepsilon)\right]^3$ and $\zeta_k = \zeta(k)$ denotes the Riemann zeta function. We also give the result for I_{10} (we use notations as in [23]) valid to $\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned} I_{10} = \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3}{D_2 D_4 D_5 D_6 D_8} = C(\varepsilon)\Bigg\{ & \frac{-1}{3\varepsilon^3} - \frac{5}{3\varepsilon^2} \\ & + \frac{1}{\varepsilon}\left(-\frac{2\pi^2}{3} - 4\right) - \frac{26}{3}\zeta_3 - \frac{7\pi^2}{3} + \frac{10}{3} \\ & + \varepsilon\left(-\frac{35\pi^4}{18} - \frac{94}{3}\zeta_3 - \pi^2 + \frac{302}{3}\right) + \varepsilon^2\Bigg(734 - \frac{76\pi^2}{3}\zeta_3 \\ & + \frac{101\pi^2}{3} - 20\zeta_3 - \frac{551}{90}\pi^4 - 462\zeta_5\Bigg)\Bigg\}. \end{aligned}$$

There are two principal checks on our solution of the recurrence relations. First, we have computed the three-loop anomalous magnetic moment of the electron and confirmed the result of Ref. [23]. Second, the actual calculation of the on-shell quark mass renormalization constant $Z_m^{\rm OS}$ has been performed in an arbitrary covariant

[3]For Abelian gauge theory, like QED, the number of basic topologies would be just four.

gauge for the gluon field. The explicit cancellation of the gauge parameter in our result for $Z_m^{\rm OS}$ is an important check of the correctness of the calculation.

Having computed the three-loop contribution to $Z_m^{\rm OS}$ in this way and using Eqs.(3,4) we obtain the three-loop relation between the $\overline{\rm MS}$ quark mass and the pole mass. Below we present the exact result only for this mass relation but we should note for future reference that $Z_m^{\rm OS}$ is easily recovered using Eq.(3). The relation between the masses is expressed in terms of color and flavor factors as

$$\begin{aligned}\overline{m}(M) = M \Bigg[& 1 - C_F\left(\frac{\alpha_s}{\pi}\right) + C_F\left(\frac{\alpha_s}{\pi}\right)^2 \Big(C_F\, d_1^{(2)} + \\ & + C_A\, d_2^{(2)} + T_R N_L\, d_3^{(2)} + T_R N_H\, d_4^{(2)}\Big) \\ & + C_F\left(\frac{\alpha_s}{\pi}\right)^3 \Big(C_F^2\, d_1^{(3)} + C_F C_A\, d_2^{(3)} + C_A^2\, d_3^{(3)} \\ & + C_F T_R N_L\, d_4^{(3)} + C_F T_R N_H\, d_5^{(3)} + C_A T_R N_L\, d_6^{(3)} \\ & + C_A T_R N_H\, d_7^{(3)} + T_R^2 N_L N_H\, d_8^{(3)} + T_R^2 N_H^2\, d_9^{(3)} \\ & + T_R^2 N_L^2\, d_{10}^{(3)}\Big) + \mathcal{O}\left(\frac{\alpha_s}{\pi}\right)^4\Bigg], \end{aligned} \tag{10}$$

where C_F and C_A are the Casimir operators of the fundamental and the adjoint representation of the color gauge group (the group is SU(3) for QCD) and T_R is the trace normalization of the fundamental representation. N_L is the number of massless quark flavors and N_H is the number of quark flavors with a pole mass equal to M. Also $\alpha_s \equiv \alpha_s^{(N_L+N_H)}(M)$ is the $\overline{\rm MS}$ strong coupling constant renormalized at the scale of the pole mass $\mu = M$ in the theory with $N_L + N_H$ active flavors[4].

Our calculation yields the following result for the coefficients $d_k^{(n)}$ in Eq. (10):

$$\begin{aligned} d_1^{(2)} &= \frac{7}{128} - \frac{3}{4}\zeta_3 + \frac{1}{2}\pi^2\log 2 - \frac{5}{16}\pi^2, \\ d_2^{(2)} &= -\frac{1111}{384} + \frac{3}{8}\zeta_3 - \frac{1}{4}\pi^2\log 2 + \frac{1}{12}\pi^2, \\ d_3^{(2)} &= \frac{71}{96} + \frac{1}{12}\pi^2, \\ d_4^{(2)} &= \frac{143}{96} - \frac{1}{6}\pi^2, \\ d_1^{(3)} &= -\frac{2969}{768} - \frac{1}{16}\pi^2\zeta_3 - \frac{81}{16}\zeta_3 + \frac{5}{8}\zeta_5 + \frac{29}{4}\pi^2\log 2 \\ &\quad + \frac{1}{2}\pi^2\log^2 2 - \frac{613}{192}\pi^2 - \frac{1}{48}\pi^4 - \frac{1}{2}\log^4 2 - 12 a_4, \\ d_2^{(3)} &= \frac{13189}{4608} - \frac{19}{16}\pi^2\zeta_3 - \frac{773}{96}\zeta_3 + \frac{45}{16}\zeta_5 - \frac{31}{72}\pi^2\log 2 \\ &\quad - \frac{31}{36}\pi^2\log^2 2 + \frac{509}{576}\pi^2 + \frac{65}{432}\pi^4 - \frac{1}{18}\log^4 2 - \frac{4}{3}a_4, \\ d_3^{(3)} &= -\frac{1322545}{124416} + \frac{51}{64}\pi^2\zeta_3 + \frac{1343}{288}\zeta_3 - \frac{65}{32}\zeta_5 \\ &\quad - \frac{115}{72}\pi^2\log 2 + \frac{11}{36}\pi^2\log^2 2 - \frac{1955}{3456}\pi^2 - \frac{179}{3456}\pi^4 \\ &\quad + \frac{11}{72}\log^4 2 + \frac{11}{3}a_4, \\ d_4^{(3)} &= \frac{1283}{576} + \frac{55}{24}\zeta_3 - \frac{11}{9}\pi^2\log 2 + \frac{2}{9}\pi^2\log^2 2 + \frac{13}{18}\pi^2 \\ &\quad - \frac{119}{2160}\pi^4 + \frac{1}{9}\log^4 2 + \frac{8}{3}a_4, \\ d_5^{(3)} &= \frac{1067}{576} - \frac{53}{24}\zeta_3 + \frac{8}{9}\pi^2\log 2 - \frac{1}{9}\pi^2\log^2 2 - \frac{85}{108}\pi^2 \\ &\quad + \frac{91}{2160}\pi^4 + \frac{1}{9}\log^4 2 + \frac{8}{3}a_4, \\ d_6^{(3)} &= \frac{70763}{15552} + \frac{89}{144}\zeta_3 + \frac{11}{18}\pi^2\log 2 - \frac{1}{9}\pi^2\log^2 2 \\ &\quad + \frac{175}{432}\pi^2 + \frac{19}{2160}\pi^4 - \frac{1}{18}\log^4 2 - \frac{4}{3}a_4, \\ d_7^{(3)} &= \frac{144959}{15552} + \frac{1}{8}\pi^2\zeta_3 - \frac{109}{144}\zeta_3 - \frac{5}{8}\zeta_5 + \frac{32}{9}\pi^2\log 2 \\ &\quad + \frac{1}{18}\pi^2\log^2 2 - \frac{449}{144}\pi^2 - \frac{43}{1080}\pi^4 - \frac{1}{18}\log^4 2 - \frac{4}{3}a_4, \\ d_8^{(3)} &= -\frac{5917}{3888} + \frac{2}{9}\zeta_3 + \frac{13}{108}\pi^2, \\ d_9^{(3)} &= -\frac{9481}{7776} + \frac{11}{18}\zeta_3 + \frac{4}{135}\pi^2, \\ d_{10}^{(3)} &= -\frac{2353}{7776} - \frac{7}{18}\zeta_3 - \frac{13}{108}\pi^2. \end{aligned}$$

where $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2) \approx 0.517479$.

For standard values of the QCD color factors $C_F = 4/3$, $C_A = 3$, $T_R = 1/2$, and assuming the number of heavy flavors N_H to be equal to one, the result reads:

$$\begin{aligned} \frac{\overline{m}(M)}{M} &= 1 - \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg(N_L\left(\frac{71}{144} + \frac{\pi^2}{18}\right) \\ &\quad - \frac{3019}{288} + \frac{1}{6}\zeta_3 - \frac{\pi^2}{9}\log 2 - \frac{\pi^2}{3}\Bigg) \\ &\quad + \left(\frac{\alpha_s}{\pi}\right)^3 \Bigg(N_L^2\left(-\frac{2353}{23328} - \frac{7}{54}\zeta_3 - \frac{13}{324}\pi^2\right) \\ &\quad + N_L\Bigg(\frac{246643}{23328} + \frac{241}{72}\zeta_3 + \frac{11}{81}\pi^2\log 2 - \frac{2}{81}\pi^2\log^2 2 \\ &\quad + \frac{967}{648}\pi^2 - \frac{61}{1944}\pi^4 - \frac{1}{81}\log^4 2 - \frac{8}{27}a_4\Bigg) \\ &\quad - \frac{9478333}{93312} + \frac{1439}{432}\zeta_3\pi^2 - \frac{61}{27}\zeta_3 - \frac{1975}{216}\zeta_5 \\ &\quad + \frac{587}{162}\pi^2\log 2 + \frac{22}{81}\pi^2\log^2 2 - \frac{644201}{38880}\pi^2 + \frac{695}{7776}\pi^4 \\ &\quad + \frac{55}{162}\log^4 2 + \frac{220}{27}a_4\Bigg). \end{aligned} \tag{11}$$

Numerically, we obtain:

$$\frac{\overline{m}(M)}{M} = 1 - \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right) + \left(\frac{\alpha_s}{\pi}\right)^2 (1.0414\, N_L - 14.3323)$$

[4]We should note that, if needed, Eq.(10) is easily expressed in terms of $\overline{m}(\mu)$, $\alpha_s(\mu)$ and $\log(\mu/M)$ by using the renormalization group equations for $\overline{m}(\mu)$ and $\alpha_s(\mu)$.

$$+\left(\frac{\alpha_s}{\pi}\right)^3\left(-0.65269\ N_L^2+26.9239\ N_L-198.7068\right). \quad (12)$$

We note that the exact three-loop $\mathcal{O}(N_L)$ and $\mathcal{O}(N_L^0)$ coefficients are within the errors of the values $27.3(7)N_L$ and 202(5) obtained in [18], nevertheless for the individual color structures $C_F C_A^2$ and $C_F^2 N_L$ the difference between the exact result and the numerical values in [18] is larger than the error bars quoted in that reference.

From Eq.(11) a relation between the pole mass M and the $\overline{m}(\overline{m})$, the $\overline{\rm MS}$ mass normalized at the scale $\mu=\overline{m}(\mu)$ is easily derived. One obtains

$$\frac{M}{\overline{m}(\overline{m})}=1+\frac{4}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right)+\left(\frac{\bar{\alpha}_s}{\pi}\right)^2(-1.0414\ N_L+13.4434)$$
$$+\left(\frac{\bar{\alpha}_s}{\pi}\right)^3\left(0.6527\ N_L^2-26.655\ N_L+190.595\right), \quad (13)$$

with $\bar{\alpha}_s\equiv\alpha_s(\overline{m})$.

It is interesting to look at these results when they are parameterized in a way, that separates the effects related to the running of the coupling constant (i.e. contributions that are related to a change of the scale μ in the coupling constant) and the remaining part. To do this in a consistent manner, we first use the decoupling relations to reexpress the $\overline{\rm MS}$ coupling constant α_s defined in the theory with N_L massless $+1$ massive flavors through the coupling, defined in the theory with N_L massless quarks only [20,21]. Using the first two coefficients of the β-function:

$$\beta_0=\frac{1}{4}\left(11-\frac{2}{3}N_L\right), \quad \beta_1=\frac{1}{16}\left(102-\frac{38}{3}N_L\right),$$

we obtain:

$$\frac{M}{\overline{m}(\overline{m})}=1+\frac{4}{3}\left(\frac{\bar{\alpha}_s}{\pi}\right)+\left(\frac{\bar{\alpha}_s}{\pi}\right)^2(6.248\ \beta_0-3.739)$$
$$+\left(\frac{\bar{\alpha}_s}{\pi}\right)^3\left(23.497\ \beta_0^2+6.248\ \beta_1+1.019\ \beta_0-29.94\right), \quad (14)$$

where $\bar{\alpha}_s$ now stands for the $\overline{\rm MS}$ coupling defined in the theory with N_L fermions only.

From the above equation one sees that the large β_0 approximation works well at $\mathcal{O}(\alpha_s^3)$ especially for $N_L=3$, but in part because of a cancellation between the conformal (i.e. β-independent) term and the term proportional to β_1. The magnitude of conformal terms is large and they exhibit a very fast growth from one order of perturbation theory to the other.

In order to check that the magnitude of conformal terms is not an artifact of our use of the $\overline{\rm MS}$ coupling constant as an expansion parameter, one can rewrite the equation for the mass ratio in terms of the V-scheme coupling $\bar{\alpha}_V=\alpha_V(\overline{m})$. To the necessary order the relation between α_V and α_s is given in Ref. [24]. Rewriting Eq.(14) in α_V, one obtains the result which does not improve as compared to the $\overline{\rm MS}$ scheme. The overall magnitude of corrections gets somewhat decreased, but conformal coefficients remain very large and grow rapidly.

This situation makes it difficult to interpret the high accuracy of the large β_0 approximation in $\mathcal{O}(\alpha_s^3)$ order simply from the assumption of a leading infrared renormalon dominance. On the one hand we clearly see that the large β_0 approximation remains extremely successful in the third order of the perturbative expansion for the pole mass. On the other hand, part of the reason for this success seems to be a cancellation between subleading β-dependent and fast growing conformal terms. Since a different physics is usually associated with these contributions it is unclear how this situation will extrapolate to the next order in the $\overline{\rm MS}$ pole mass relation. We refrain from drawing a definite conclusion on this point at the moment.

Since exact calculations in even higher orders of perturbation theory seem formidably complicated, it is perhaps reasonable to look for other approximation schemes which, as compared to the large β_0 approximation, will preserve in a better way the field theoretical structure of QCD. One possibility is the large N_c approximation [25]. Its attractive feature is that only planar diagrams contribute to leading order in N_c and this significantly reduces the technical burden of the calculations. Since in realistic QCD we are interested in the applications where $N_c\sim N_L$, one can go a step further and consider a situation where both the number of colors N_c and the number of light fermions N_L are considered to be large numbers. The correction to this approximation is expected to be of the order $\mathcal{O}(N_c^{-2})$ which for $N_c=3$ is about 10 percent[5]. The result for the $\overline{\rm MS}$ mass in this approximation reads:

$$\left[\frac{\overline{m}(M)}{M}\right]_{N_c}=\Bigg(1-1.5\left(\frac{\alpha_s}{\pi}\right)+\left(\frac{\alpha_s}{\pi}\right)^2(1.17\ N_L-16.13)$$
$$+\left(\frac{\alpha_s}{\pi}\right)^3\left(-0.73\ N_L^2+30.41\ N_L-222.84\right)\Bigg). \quad (15)$$

Comparing this equation with Eq. (12) one sees that the large N_c approximation works with the expected accuracy and hence might serve as a useful starting point for the analysis at an even higher order.

III. THE THREE-LOOP ANALYSIS OF SEMILEPTONIC B DECAYS

Let us now turn to the analysis of the semileptonic decay process $B\to X_u e\nu_e$. We want to investigate whether various existing estimates of the $\mathcal{O}(\alpha_s^3)$ correction to this process are compatible with the notion of an improved quality of perturbative series at low orders when the pole quark mass is eliminated in favor of a properly defined

[5]For contributions of massive fermion loops the accuracy of the approximation is formally $\mathcal{O}(N_c^{-1})$ but due to the mass of these fermions these contributions are usually small.

low-scale short-distance mass. Some of these estimates refer explicitly to the $\overline{\rm MS}$ mass and this investigation requires the use of the three-loop quark mass relation Eq. (13).

Recall, that the Heavy Quark Expansion demonstrates that the non-perturbative effects are small in this process [26] and an important issue is the accurate calculation of the perturbative corrections to the quark level decay $b \to ue\nu_e$. To $\mathcal{O}(\alpha_s^2)$ such a calculation has been performed in Ref. [27]. Recently the three-loop correction to that process has been estimated in Ref. [28] using asymptotic Pade approximants. It is unclear to us at the moment how accurate that estimate is, but we will take the number quoted in Ref. [28] as a reference point. For five active flavors we write:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{ub}|^2 \overline{m}(\overline{m})^5}{192\pi^3} \left\{ 1 + 4.25 \left(\frac{\bar{\alpha}_s^{(5)}}{\pi} \right) + 26.87 \left(\frac{\bar{\alpha}_s^{(5)}}{\pi} \right)^2 + (200 + \Delta) \left(\frac{\bar{\alpha}_s^{(5)}}{\pi} \right)^3 \right\} . \quad (16)$$

The central value of the third order coefficient is due to Ref. [28] and Δ parameterizes possible uncertainties in this estimate. For the chosen normalization point $\mu = \overline{m}$, one sees a rapid increase in the coefficients of the series in Eq.(16). The reason for that is thought to be a poor choice of the renormalization scale, since a typical energy release in B decays is of the order $m_b/5$ rather than m_b. Since at very low scales the logarithmic running of the quark mass is considered unphysical, it was suggested to use other short distance masses which have a more transparent meaning at a low normalization point. In this paper we would like to check the behavior of perturbation series once the decay width is expressed in terms of the so-called $1S$ mass. Strictly speaking, the $1S$ mass introduced in Ref. [12] is not a truly short distance quantity and its usefulness for B decays when both perturbative and non-perturbative corrections are taken into account remains to be proven. Nevertheless, we use it here because its relation to the pole mass is known to a sufficiently high order in the perturbative expansion. It was suggested that the correct way to incorporate the $1S$ mass into semileptonic B decays is the Υ expansion [29] and we will denote different terms in this expansion by a formal parameter ϵ (which is eventually put to one). We use the relation between the $\overline{\rm MS}$ mass and the pole mass to rewrite Eq.(16) in terms of the $1S$ mass (via its relation to the pole mass) and check if the result is consistent with an improved behavior of the perturbation series. Below we consider four possible scenarios.

i) The large β_0 estimate of the three-loop coefficient of $\Gamma_{\rm sl}$ is accurate[6]. Using the mass relation, derived in this paper, we find that this requires $\Delta \approx -120$ in Eq.(16). We then rewrite the decay rate in terms of the $1S$ mass and obtain for $\bar{\alpha}_s^{(4)} = 0.22$:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{ub}|^2 m_{1S}{}^5}{192\pi^3} \left(1 - 0.115\epsilon - 0.031\epsilon^2 - 0.033\epsilon^3 \right) .$$

ii) The pattern of the n-th loop coefficient in Eq.(16) is well approximated by 5^n (see Ref. [31]). For the three-loop coefficient in Eq.(16) this implies $\Delta \approx -75$. In this case the large β_0 estimate is about 10 per cent above the true value. Expressed through the $1S$ mass, the decay rate for $\bar{\alpha}_s^{(4)} = 0.22$ reads:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{ub}|^2 m_{1S}{}^5}{192\pi^3} \left(1 - 0.115\epsilon - 0.031\epsilon^2 - 0.017\epsilon^3 \right) .$$

iii) The estimate of Ref. [28] is accurate. The exact result is then smaller by a factor 0.7 than the large β_0 approximation. Rewriting the decay rate through the $1S$ mass and using $\bar{\alpha}_s^{(4)} = 0.22$ we obtain:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{ub}|^2 m_{1S}{}^5}{192\pi^3} \left(1 - 0.115\epsilon - 0.031\epsilon^2 - 0.01\epsilon^3 \right) .$$

iv) At the two loop level the exact result is about a factor 0.8 smaller than the large β_0 approximation. Let us imagine that in three loops the exact result is about one half of the large β_0 approximation. This implies $\Delta \approx 60$. For $\bar{\alpha}_s^{(4)} = 0.22$ we obtain:

$$\Gamma_{\rm sl} = \frac{G_F^2 |V_{ub}|^2 m_{1S}{}^5}{192\pi^3} \left(1 - 0.115\epsilon - 0.031\epsilon^2 + 0.03\epsilon^3 \right) .$$

We see therefore that there is a window for the parameter Δ where different approximations to the full result work fairly well and the parameterization of the decay width through the $1S$ mass seems to succeed in making perturbative series convergent. However, there is one unpleasant feature. The magnitude of the $\mathcal{O}(\epsilon^3)$ term in Υ expansion strongly depends on the value of the strong coupling constant and this is because terms which scale differently in α_s contribute to a given term in Υ expansion[7]. In general, this dependence is quite strong. For example, by using $\bar{\alpha}_s = 0.25$ we would have obtained the 0.05 as a coefficient of the ϵ^3 term in the fourth example.

[6] When we refer to the results of "the large β_0 approximation" in this context, we have in mind the expression for $\Gamma_{\rm sl}$ written in terms of the pole mass. The predictions for $\Gamma_{\rm sl}$ in the large β_0 approximation can be found in Ref. [30].

[7] In present example, the highest power of α_s in the ϵ^3 terms is six and this contribution comes from the leading order difference between the $1S$ and the pole mass.

IV. CONCLUSIONS

The main result of this letter, an analytic three-loop relation between the $\overline{\mathrm{MS}}$ and the pole masses, is given by Eq.(11). It provides a first step towards a more precise analysis of QCD effects in different low scale short distance processes.

Although this result is in good agreement with the large β_0 approximation, we have argued that it is difficult to understand the structure of subleading terms. Nevertheless, our analysis of semileptonic B decay $B \to X_u e\nu$ shows that it is likely that the general physical picture of improving a badly behaved perturbative series at low orders by eliminating the pole quark mass remains valid also in the three-loop order. However, to answer this question unambiguously, one would have to perform a three-loop QCD calculation of semileptonic decays of a heavy quark and this seems to be a very non-trivial task at present. It may be that the large N_c approximation will provide enough technical simplifications to be feasible beyond the presently known perturbative orders and at the same time be accurate enough to be meaningful.

ACKNOWLEDGMENTS

We are grateful to D. Broadhurst, J.H. Kühn, S. Larin and N. Uraltsev for useful comments. This research was supported in part by the United States Department of Energy, contract DE-AC03-76SF00515, by BMBF under grant number BMBF-057KA92P, by Graduiertenkolleg "Teilchenphysik" at the University of Karlsruhe and by the DFG Forschergruppe "Quantenfeldtheorie, Computeralgebra und Monte-Carlo-Simulation".

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