Relating the Quark Mass defined in the "Regularization Invariant" Scheme to the $\overline{\rm MS}$ Mass at $\mathcal{O}(\alpha_s^3)$

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We report on the analytical calculation of the $\mathcal{O}(\alpha_s^3)$ conversion factor between the $\overline{\rm MS}$ quark mass and the one defined in the so-called "Regularization Invariant" scheme.

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1. Introduction

Although the quark masses are fundamental parameters of the QCD Lagrangian, their relation to measurable physical quantities is not direct. They depend on the renormalization scheme and, within a given one, on the renormalization scale μ.

In the realm of perturbative QCD the most frequently used mass definition is the so-called short distance $\overline{\rm MS}$ mass, based on the $\overline{\rm MS}$-scheme [1, 2]. Unfortunately it is difficult to obtain precise information on the quark masses from pQCD, as the mass dependence of it's predictions is relatively weak.

One possibility to obtain such information are lattice QCD calculations, which provide a direct way to determine quark masses from first principles (for recent discussions see [5, 6, 7, 8, 9]). The resulting quark mass is the (short distance) bare lattice quark mass. A scheme that is directly accessible in lattice calculations is the "Regularization Independent" scheme, which has been used in some recent lattice calculations and was proposed in [4].

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Relating lattice quark masses to those defined in a continuum perturbative scheme as the $\overline{\rm MS}$ one requires the calculation of the corresponding renormalization constants. These constants can be defined and computed only *perturbatively.* The conversion factor for the mass defined in the RI scheme and the $\overline{\rm MS}$ scheme is now known at next-to-next-to-leading order (NNLO) from [10] and happens to be numerically significant. This makes mandatory to know the NNNLO $O(\alpha_s^3)$ term in the conversion factor.

In this work we report on the calculation of this term. It turns out that the size of this term is comparable to the previous one at a renormalization scale of 2 GeV — the typical scale currently used in lattice calculations of the light quark masses.

2. Scheme Dependence of the Quark Mass

To calculate the conversion factors, we start with the bare quark propagator (for simplicity we stick to the Landau gauge and do not explicitly display the gauge dependence)

$$S_0(q,\alpha_s^0,m_0) = \mathrm{i}\int \mathrm{d}x \mathrm{e}^{\mathrm{i}qx}\langle \mathrm{T}[\psi_0(x)\bar{\psi}_0(0)]\rangle = (m_0 - \not{q} - \Sigma_0)^{-1} \qquad (1)$$

with the quark mass operator Σ_0 being conveniently decomposed into Lorentz invariant structures according to $\Sigma_0 = \not{q}\Sigma_V^0 + m_0\Sigma_S^0$. Here m_0 and ψ_0 are the bare quark mass and field respectively and $a_s^0 \equiv \alpha_s^0/\pi = g^2/(4\pi^2)$, where g is the bare QCD gauge coupling.

Higher order corrections to physical quantities in pQCD will give finite results only after regularization and reparameterization of all parameters and fields of the theory. Moreover, Greens functions as (1) also need to be renormalized. To be precise we assume that (1) is dimensionally regulated by going to non-integer values of the space-time dimension $D = 4 - 2\epsilon$ [11, 12]. Then the $\overline{\rm MS}$ renormalized Green function (1) reads:

$$S(q,\alpha_s,m,\mu) = (m - \not{q} - \Sigma)^{-1} = Z_2^{-1} S_0(q,\alpha_s^0,m_0)|_{m_0 = Z_m m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha \alpha_s} \qquad (2)$$

where $\psi = Z_2^{-1/2}\psi_0$ is the renormalized quark field and the 't Hooft mass parameter μ is a scale at which the renormalized quark mass is defined. The renormalization constants Z_2, Z_α and Z_m are series of the generic form

$$Z_? = 1 + \sum_{i>0} Z_?^{(i)}\frac{1}{\epsilon^i}, \quad Z_?^{(i)} = \sum_{j\geq i} Z_?^{(i,j)}\left(\frac{\alpha_s}{\pi}\right)^j, \quad ? = 2, \alpha, m. \qquad (3)$$

The quark propagator renormalized according to a different subtraction procedure reads (parameters marked with a prime belong to the other scheme)

$$S(q, \alpha_s', m', \mu) = \frac{1}{m' - \not{q} - \Sigma'} = (Z_2')^{-1} S_0(q, \alpha_s^0, m_0)|_{m_0 = Z_m' m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha' \alpha_s}, \tag{4}$$

where without essential loss of generality we have set $\mu' = \mu$. The finiteness of the renormalized fields and parameters in both schemes implies that, within perturbation theory, the relation between them is uniquely:

$$m = \frac{Z_m'}{Z_m} \cdot m' = C_m \cdot m', \;\; \psi = \sqrt{\frac{Z_2'}{Z_2}} \cdot \psi' = \sqrt{C_2} \cdot \psi', \tag{5}$$

with the "conversion functions" being themselves *finite* series in α_s', i.e.

$$C_? \equiv 1 + \sum_{i>0} C_?^{(i)} \left(\frac{\alpha_s'}{\pi}\right)^i \qquad , \qquad ? = m, 2. \tag{6}$$

In general the coefficients $C_?^i$ may depend on the ratio m'/μ. If such a dependence is absent the corresponding scheme is called a "mass independent" one. In what follows, we will assume that the function C_α is known and, thus, will deal with series of the type (6) in terms of the $\overline{\mathrm{MS}}$ α_s

From eqs. (1) it is easy to see that

$$C_2 \cdot (1 + \Sigma_V) = 1 + \Sigma_V' \quad , \quad C_2 \cdot C_m \cdot (1 - \Sigma_S) = 1 - \Sigma_S' \tag{7}$$

These equations together with renormalization conditions for the non-$\overline{\mathrm{MS}}$ scheme then provide the necessary information to determine the conversion factors C_m and C_2, given the $\overline{\mathrm{MS}}$ renormalized Σ_V and Σ_S.

A mass independent MOM[1] scheme has recently been suggested in [4] under the name of RI ("Regularization Invariant") and is defined by[2]:

$$\lim_{m\to 0} \tfrac{1}{48} Tr\left[\gamma_\mu \frac{\partial\left(\not{q}(1+\Sigma_V^{\mathrm{RI}})\right)}{\partial q_\mu}\right]_{q^2=-\mu^2} = 1\,,\; \lim_{m\to 0} \tfrac{1}{12} Tr\left[1-\Sigma_S^{\mathrm{RI}}\right]_{q^2=-\mu^2} = 1 \tag{8}$$

From these we get the conversion constants ($\ell = log(-\frac{q^2}{\mu^2})$):

$$C_2^{\mathrm{RI}} = \left[1+\Sigma_V + \tfrac{1}{2}\cdot\frac{\partial \Sigma_V(\ell)}{\partial \ell}\right]^{-1}_{\substack{q^2=-\mu^2\\ m=0}}\,,\; C_m^{\mathrm{RI}} = \left[\frac{1+\Sigma_V + \frac{1}{2}\cdot\frac{\partial \Sigma_V(\ell)}{\partial \ell}}{1-\Sigma_S}\right]_{\substack{q^2=-\mu^2\\ m=0}} \tag{9}$$

[1] The so-called momentum subtraction schemes require the values of Green Functions with fixed μ dependent external momentum configurations to be fixed

[2] Traces are to be taken over color, Lorentz and Dirac indices

3. Three Loop $\overline{\mathrm{MS}}$ Quark Propagator

We have analytically computed the functions Σ_V and Σ_S in the massless limit to order α_s^3. In addition, we have calculated the first 6 (3 for the 3-loop case) terms in a small mass expansion of these functions. The calculation was making intensive use of computer algebra programs. In particular, we have used `QGRAF` [14] for the generation of diagrams and `LMP` [17] for the asymptotic expansions (for an introduction see [3]). The resulting massless propagator diagrams and massive tadpole diagrams have been evaluated with the form packages `MINCER` [15] and `MATAD` [16]. Up to 2 loops the analytical mass dependence of the functions Σ_V and Σ_S are known [18]. These results have been used as cross checks for our results and we found full agreement in numerical evaluation and small and large mass expansions of their result. The full results of these functions are published in [13].

4. Results

Our result for the conversion function for the RI mass to the $\overline{\mathrm{MS}}$ mass reads[3] as functions of n_f:

$$
\begin{aligned}
C_m^{\mathrm{RI}} = 1 &+ \frac{\alpha_s}{4\pi}\left[-\frac{16}{3}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{1990}{9} + \frac{152}{3}\,\zeta_3 + \frac{89}{9}\,n_f\right] \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^3\left[-\frac{6663911}{648} + \frac{408007}{108}\,\zeta_3 - \frac{2960}{9}\,\zeta_5 + \frac{236650}{243}\,n_f \right. \\
&\left. - \frac{4936}{27}\,\zeta_3\,n_f + \frac{80}{3}\,\zeta_4\,n_f - \frac{8918}{729}\,n_f^2 - \frac{32}{27}\,\zeta_3\,n_f^2\right].
\end{aligned}
$$

At a scale $\mu = 2$ GeV and $n_f = 4$, the numerical contributions of the leading order to NNNLO terms are as follows (with $\alpha_s(2\,\mathrm{GeV})/\pi = 0.1$)

$$
C_m^{\mathrm{RI}} = 1. - 0.133333 - 0.0754071 - 0.0495357
$$

One observes that the sizes of the NNLO and NNNLO contributions to C_m^{RI} at this scale amount to about 7.5% and 5% respectively. This shows that the NNNLO term is numerically significant and should be taken into account when transforming the RI quark masses to the $\overline{\mathrm{MS}}$ ones. Indeed, the size of the NNNLO term makes the applicability of pQCD at this scale doubtful. For a more elaborate discussion on the result see [13].

[3] ζ_x are the values of Riemann's Zeta function, note that C_m^{RI} is gauge independent

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REFERENCES

[1] G.'t Hooft, *Nucl. Phys.* **B61** (1973) 455.

[2] W.A. Bardeen, A.J. Buras, D.W. Duke and T. Muta, *Phys. Rev.* **D18** (1978) 3998.

[3] Robert Harlander, these proceedings.

[4] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, *Nucl. Phys.* **B445** (1995) 81.

[5] C.A. Dominguez, L. Pirovano and K. Schilcher, *Phys. Lett.* **B425** (1997) 193.

[6] S. Narison, `hep-ph/9712386`.

[7] K.G. Chetyrkin, J.H. Kuhn and A.A. Pivovarov, *Nucl. Phys.* **B533** (1998) 473.

[8] R.D. Kenway, `hep-lat/9810054`.

[9] V. Lubicz, `hep-ph/9809417`.

[10] E. Franco and V. Lubicz, *Nucl. Phys.* **B531** (1998) 641.

[11] G.'t Hooft and M. Veltman, *Nucl. Phys.* **B44** (1972) 189.

[12] C.G. Bollini and J.J. Giambiagi, *Phys. Let.* **40B** (1972) 566;
G.M. Cicuta and E. Montaldi, *Nuovo Cim. Lett.* **4** (1972) 329;
J.F. Ashmore, *Nuovo Cim. Lett.* **4** (1972) 289.

[13] K. G. Chetyrkin and A. Rétey, `hep-ph/9910332`.

[14] P. Nogueira, *J. Comp. Phys.* **105** (1993) 279.

[15] S.A. Larin, F.V. Tkachov, and J.A.M. Vermaseren, Rep. No. NIKHEF-H/91-18 (Amsterdam, 1991).

[16] M. Steinhauser, PhD thesis, Shaker Verlag, Aachen, 1996, ISBN 3-8265-1680-X.

[17] R. Harlander, PhD thesis, Shaker Verlag, Aachen 1998, ISBN 3-8265-4545-1.

[18] J. Fleischer, F. Jegerlehner, O.V. Tarasov and O.L. Veretin, *Nucl. Phys.* **B539** (1998) 671.