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Renormalization and Running of Quark Mass and Field in the Regularization Invariant and $\overline{\rm MS}$ Schemes at Three and Four Loops

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Abstract

We derive explicit transformation formulae relating the renormalized quark mass and field as defined in the $\overline{\rm MS}$-scheme with the corresponding quantities defined in any other scheme. By analytically computing the three-loop quark propagator in the high-energy limit (that is keeping only massless terms and terms of first order in the quark mass) we find the NNNLO conversion factors transforming the $\overline{\rm MS}$ quark mass and the renormalized quark field to those defined in a "Regularization Invariant" (RI) scheme which is more suitable for lattice QCD calculations. The NNNLO contribution in the mass conversion factor turns out to be large and comparable to the previous NNLO contribution at a scale of 2 GeV — the typical normalization scale employed in lattice simulations. Thus, in order to get a precise prediction for the $\overline{\rm MS}$ masses of the light quarks from lattice calculations the latter should use somewhat higher scale of around, say, 3 GeV where the (apparent) convergence of the perturbative series for the mass conversion factor is better.

We also compute two more terms in the high-energy expansion of the $\overline{\rm MS}$ renormalized quark propagator. The result is then used to discuss the uncertainty caused by the use of the high energy limit in determining the $\overline{\rm MS}$ mass of the charmed quark. Finally, as a by-product of our calculations we determine the four-loop anomalous dimensions of quark mass and field in the Regularization Invariant scheme.

1 Introduction

Quark masses are fundamental parameters of the QCD Lagrangian. Nevertheless, their relation to measurable physical quantities is not direct: the masses depend on the renormalization scheme and, within a given one, on the renormalization scale μ.

In the realm of pQCD the definition which is most often used is based on the $\overline{\rm MS}$-scheme [1, 2] which leads to the so-called short-distance $\overline{\rm MS}$ mass. Such a definition is of great convenience for dealing with mass-dependent inclusive physical observables dominated by short distances (for a review see [3]). Unfortunately it is usually difficult to get precise information about the quark masses from predictions from these considerations, as their mass dependence is relatively weak.

To determine the absolute values of quark masses, one necessarily has to rely on the methods which incorporate the features of nonperturbative QCD. So far, the only two methods which are based on

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QCD from first principles are QCD sum rules and Lattice QCD (for recent discussions see e.g. [4]–[13]). Rather accurate determinations of the ratios of various quarks masses can be obtained within Chiral Perturbation Theory [14].

Lattice QCD provides a direct way to determine quark masses from first principles. Unlike QCD sum rules it does not require model assumptions. It is possible to carry out the systematic improvement of Lattice QCD so that all the discretization errors proportional to the lattice spacing are eliminated (a comprehensible review is given in [15]). The resulting quark mass is the (short distance) bare lattice quark mass. The matching of the lattice quark masses to those defined in a continuum perturbative scheme requires the calculation of the corresponding multiplicative renormalization constants. In the RI scheme [16] the renormalization conditions are applied to amputated Green functions in Landau gauge, setting them equal to their tree-level values. This allows the non-perturbative calculation of the renormalization constants. An alternative to the RI approach is the Schroedinger functional scheme (SF) which was used in [17, 18].

An impressive number of various lattice determinations of quark masses has recently been performed (see Refs [19]–[31]).

Once the RI quark masses are determined from lattice calculations they can be related to the $\overline{\mathrm{MS}}$ mass by a corresponding conversion factor. By necessity this factor can be defined and, hence, computed only *perturbatively*. The conversion factor is presently known at next-to-next-to-leading order (NNLO) from [32]. The NNLO contribution happens to be numerically significant. This makes mandatory to know the NNNLO $O(\alpha_s^3)$ term in the conversion factor.

In the present article we describe the calculation of this term. It turns out that the size of the newly computed term is comparable to the previous one at a renormalization scale of 2 GeV — the typical scale currently used in lattice calculations of the light quark masses. This means that perturbation theory can not be used for a precise conversion of the presently available RI quark masses to the $\overline{\mathrm{MS}}$ ones. A simple analysis shows that the convergence gets much better if the scale is increased to, say, 3 GeV. Thus, once the lattice calculations produce the RI quark masses at this scale our formulas will allow an accurate conversion to the $\overline{\mathrm{MS}}$ masses at the same scale.

The article is organized as follows. In section 2 we discuss the scheme dependence of the quark field and mass and present a general procedure to find corresponding conversion factors from one scheme to another. The general technique is illustrated by constructing the conversion factors between the $\overline{\mathrm{MS}}$ and the RI scheme. In section 3, we present the first few terms of the small mass expansion of the three-loop quark-propagator in the $\overline{\mathrm{MS}}$ scheme. In section 4 we first present the conversion functions for the quark mass and field, and then investigate the validity of the massless approximation for these functions. Then we use these results to calculate the anomalous dimensions for the quark mass and field in the RI scheme, the so-called RG invariant mass $\hat{m}$. The final section is devoted to conclusions.

In Appendices A and B we display our results for the small mass expansion of the fermion propagator and the various conversion factors with their full dependence on the group theoretical factors C_F, C_A and T. In Appendix C the four loop anomalous dimensions of the quark mass and field are listed for the case of a $SU(N)$ gauge group.

2 Scheme dependence of quark mass and field

2.1 Generalities

We start by considering the bare quark propagator (for simplicity we stick to the Landau gauge in this section and, thus, do not explicitly display the gauge dependence)

$$S_0(q,\alpha_s^0,m_0) = i\int dx e^{iqx}\langle T[\psi_0(x)\bar{\psi}_0(0)]\rangle = \frac{1}{m_0 - \not{q} - \Sigma_0}, \tag{1}$$

with the quark mass operator Σ_0 being conveniently decomposed into Lorentz invariant structures according to

$$\Sigma_0 = \not{q}\Sigma_V^0 + m_0\Sigma_S^0, , \tag{2}$$

where m_0 and ψ_0 are the bare quark mass and field, respectively. Additionally we are using the common shortcuts for the coupling constant throughout this article:

$$a_s \equiv \frac{\alpha_s}{\pi} = \frac{g^2}{4\pi^2}\,,$$

g being the bare QCD gauge coupling. To be precise we assume that (1) is dimensionally regulated by going to non-integer values of the space-time dimension $D = 4 - 2\epsilon$ [33, 34]. The $\overline{\mathrm{MS}}$ renormalized counterpart of the Green function (1) reads

$$\begin{aligned} S(q,\alpha_s,m,\mu) &= i\int dx e^{iqx}\langle T[\psi(x)\bar\psi(0)]\rangle = \frac{1}{m - \not{q} - \Sigma} \\ &= Z_2^{-1} S_0(q,\alpha_s^0,m_0)|_{m_0 = Z_m m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha \alpha_s}\,, \end{aligned} \tag{3}$$

where the renormalized quark field is

$$\psi = Z_2^{-1/2}\psi_0$$

and the 'Hooft mass parameter μ is the scale at which the renormalized quark mass is defined. The renormalization constants Z_2, Z_α and Z_m are series of the generic form

$$Z_? = 1 + \sum_{i>0} Z_?^{(i)}\frac{1}{\epsilon^i}, \quad , \; Z_?^{(i)} = \sum_{j\geq i} Z_?^{(i,j)}\left(\frac{\alpha_s}{\pi}\right)^j. \tag{4}$$

A general theorem (first rigorously proven for minimal subtractions in [35, 36]) states that there is a unique choice of renormalization constants of the form (4) which makes the propagator[1] finite in the limit of $D \to 4$.

The independence of the bare coupling constant, mass and quark field on μ leads in the standard way to the following renormalization group equations for their renormalized counterparts

$$\mu^2\frac{d}{d\mu^2}\left[\frac{\alpha_s(\mu)}{\pi}\right]|_{g_0,m_0} = \beta(\alpha_s) \equiv -\sum_{i\geq 0}\beta_i\left(\frac{\alpha_s}{\pi}\right)^{i+2} = \sum_{i>0}(i)Z_\alpha^{(1,i)}, \tag{5}$$

$$\mu^2\frac{d}{d\mu^2}\overline{m}(\mu)|_{g_0,m_0} = m(\mu)\gamma_m(\alpha_s) \equiv -\overline{m}\sum_{i\geq 0}\gamma_m^{(i)}\left(\frac{\alpha_s}{\pi}\right)^{i+1} = m(\mu)a_s\frac{\partial Z_m^{(1)}}{\partial a_s} \tag{6}$$

and

$$2\mu^2\frac{d}{d\mu^2}\psi(\mu)|_{\psi_0,g_0,m_0} = \psi(\mu)\gamma_2(\alpha_s) \equiv \psi(\mu)\sum_{i\geq 0}-\gamma_2^{(i)}\left(\frac{\alpha_s}{\pi}\right)^{i+1} = \psi(\mu)a_s\frac{\partial Z_2^{(1)}}{\partial a_s}. \tag{7}$$

Now let us consider the quark propagator renormalized according to a different subtraction procedure. Marking with a prime parameters of the second scheme one can write

$$\begin{aligned} S(q,\alpha_s',m',\mu) &= i\int dx e^{iqx}\langle T[\psi'(x)\bar\psi'(0)]\rangle = \frac{1}{m' - \not{q} - \Sigma'} \\ &= (Z_2')^{-1} S_0(q,\alpha_s^0,m_0)|_{m_0 = Z_m' m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha' \alpha_s}\,, \end{aligned} \tag{8}$$

where without essential loss of generality we have set $\mu' = \mu$. The finiteness of the renormalized fields and parameters in both schemes implies that, within the perturbation theory framework, the relation between them can be uniquely described as follows

$$m = C_m \cdot m' \tag{9}$$

$$\psi = \sqrt{C_2}\cdot\psi', \tag{10}$$

[1]In fact all Green functions if proper renormalization constants for gluon and ghost fields are introduced.

with the "conversion functions" being themselves *finite* series in α'_s, i.e.

$$C_? \equiv 1 + \sum_{i>0} C_?^{(i)} \left(\frac{\alpha'_s}{\pi}\right)^i \tag{11}$$

for $? = m$ or 2.

Note that in general the coefficients $C_?^i$ may depend on the ratio m'/μ. If such a dependence is absent then the corresponding subtraction scheme is referred to as a "mass independent" one. In what follows we mainly limit ourselves to considering this latter case. In addition, being only interested in the conversion functions C_2 and C_m, we will assume that the function C_α has already been determined and, thus, will deal with the following representation of C_2 and C_m in terms of the $\overline{\rm MS}$ coupling constant α_s:

$$C_? \equiv 1 + \sum_{i>0} C_?^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i. \tag{12}$$

The running of m' and ψ' is governed by the corresponding anomalous dimensions $\gamma_m(a_s)$ and $\gamma_2(a_s)$. A direct use of Eqs. (9,10) gives

$$\gamma'_m = \gamma_m - \beta \frac{\partial}{\partial a_s} \ln C_m, \tag{13}$$

$$\gamma'_2 = \gamma_2 - \beta \frac{\partial}{\partial a_s} \ln C_2. \tag{14}$$

At last, from Eq. (1) it is easy to see that

$$S(q) = C_2 \cdot F'(q) = \frac{C_2}{m'(1-\Sigma'_S) - \not{q}(1+\Sigma'_V)} \tag{15}$$

or, equivalently,

$$C_2 \cdot (1+\Sigma_V) = 1 + \Sigma'_V \tag{16}$$

$$C_2 \cdot C_m \cdot (1-\Sigma_S) = 1 - \Sigma'_S \tag{17}$$

The renormalization conditions for the non-$\overline{\rm MS}$ scheme should then be used to provide the necessary information about the right hand side to calculate the conversion factors C_m and C_2 given the $\overline{\rm MS}$ renormalized Σ_V and Σ_S.

2.2 Regularization Invariant scheme versus $\overline{\rm MS}$

The $\overline{\rm MS}$ subtraction scheme is intimately connected to dimensional regularization and, thus, can not be directly used with other regularizations, including the lattice one. In addition, the physical meaning of its normalization parameter μ is not transparent and leads to the well-known ambiguities when considering the decoupling of heavy particles.

It is well-known that the above shortcomings are absent for a wide class of so-called momentum subtraction (MOM) schemes[2]. The MOM schemes require the values of properly chosen Green functions with predefined μ dependent configurations of external momenta to be fixed (usually to their tree values) independently on the considered order. Practical calculations can then be performed with any regulator (or even without it in the regulator-free approach of [37, 38]). A shortcoming of MOM schemes is that they are in general not mass-independent which leads to a complicated running of coupling constant(s) and mass(es).

A general analysis of the problem of constructing of mass-independent subtraction schemes was performed long ago in [39]. Following essentially Weinberg's ideas, a specific example of a mass

[2]In a sense the oldest subtraction scheme — the on-shell one for QED — can also be considered as an example of a MOM scheme.

independent MOM scheme for QCD has recently been considered in [16] under the name of RI ("Regularization Invariant") scheme (see also [40]). The corresponding renormalization conditions for $\Sigma_S^{\rm RI}$ and $\Sigma_V^{\rm RI}$ read

$$
\begin{aligned}
\lim_{m\to 0} \tfrac{1}{48} Tr \left[\gamma_\mu \frac{\partial\left(\not{q}(1+\Sigma_V^{\rm RI}\ ) \right)}{\partial q_\mu} \right]_{q^2=-\mu^2} &= 1 \\
\lim_{m\to 0} \tfrac{1}{12} Tr \left[1 - \Sigma_S^{\rm RI}\ \right]_{q^2=-\mu^2} &= 1,
\end{aligned}
\tag{18}
$$

where the trace is to be taken over Dirac, Lorentz and colour indices. Note that the zero mass limit in (18) means that both $\Sigma_V^{\rm RI}$ and $\Sigma_S^{\rm RI}$ are effectively *massless* functions only depending on the QCD coupling constant, the normalization point μ and q^2. This also implies that it is also sufficient to compute the $\overline{\rm MS}$ functions Σ_V and Σ_S in massless QCD when computing the conversion factors from relations (16) and (17).

Application of the renormalization conditions (18) to the conversion formulae (17) and (17) leads to equations that can simply be solved for $C_m^{\rm RI}$ and $C_2^{\rm RI}$. All the dependence of Σ_V on q^2 is of the form of $\ell = log(-\frac{q^2}{\mu^2})$, which simplifies the trace and derivative w.r.t. q_μ and leads to

$$
C_2^{\rm RI} = \left[1 + \Sigma_V + \frac{1}{2}\cdot\frac{\partial\Sigma_V(\ell)}{\partial\ell} \right]^{-1}_{q^2=-\mu^2,\, m=0} \tag{19}
$$

$$
C_m^{\rm RI} = \left[\frac{1+\Sigma_V + \frac{1}{2}\cdot\frac{\partial\Sigma_V(\ell)}{\partial\ell}}{1-\Sigma_S} \right]_{q^2=-\mu^2\ m=0} . \tag{20}
$$

Another quark field renormalization that is useful in numerical lattice simulations is defined by the condition

$$
\lim_{m\to 0} \frac{1}{12} Tr \left[1 + \Sigma_V^{\rm RI'} \right]_{q^2=-\mu^2} = 1\,, \tag{21}
$$

which results in the even simpler conversion factors

$$
C_2^{\rm RI'} = \left[1 + \Sigma_V \right]^{-1}_{q^2=-\mu^2,\, m=0} \tag{22}
$$

$$
C_m^{\rm RI'} = \left[\frac{1+\Sigma_V}{1-\Sigma_S} \right]_{q^2=-\mu^2,\, m=0} . \tag{23}
$$

Using these equations all conversion factors can easily be obtained, once the $\overline{\rm MS}$ renormalized expressions Σ_S and Σ_V are known.

It should be noted that in practical lattice calculations the massless limit on the left hand side of (18) and (21) is implemented by choosing $\mu \gg m$. On the other hand, μ should be much less than the inverse lattice spacing $1/a$. Typically μ is taken around 2 GeV. This means that lattice determinations do not lead directly to the RI quark mass but rather to the mass in a different, mass-dependent, scheme. The difference between both schemes can be numerically non-negligible for the case of the charmed quark.

Thus, for both RI and RI' it is suggestive to introduce their mass-dependent counterparts MOM and MOM' as defined by the same Eqs. (18) and (21) but without the $m \to 0$ requirement. The corresponding conversion factors to the $\overline{\rm MS}$ scheme read

$$
C_2^{MOM} = \left[1 + \Sigma_V + \frac{1}{2}\cdot q^2 \frac{\partial\Sigma_V}{\partial q^2} \right]^{-1}_{q^2=-\mu^2} \tag{24}
$$

$$
C_m^{MOM} = \left[\frac{1+\Sigma_V + \frac{1}{2}\cdot q^2\frac{\partial\Sigma_V}{\partial q^2}}{1-\Sigma_S} \right]_{q^2=-\mu^2} \tag{25}
$$

and

$$C_2^{MOM'} = [1+\Sigma_V]^{-1}_{q^2=-\mu^2} \tag{26}$$

$$C_m^{MOM'} = \left[\frac{1+\Sigma_V}{1-\Sigma_S}\right]_{q^2=-\mu^2} . \tag{27}$$

respectively.

3 Three-loop quark propagator in $\overline{\text{MS}}$-scheme

To find the conversion factors for MOM and MOM' one needs to compute the functions Σ_V and Σ_S including their full mass dependence. A full analytical result at two-loop level has been obtained only recently in [41]. An extension of this calculation up to three-loops is out of reach of present calculational technologies. Fortunately for the RI and RI' schemes one effectively only needs to compute *massless* three-loop diagrams – a problem which in principle was solved long ago in [42]. A promising approach to recover the full mass dependence of the quark propagator seems to be to employ an expansion in (m^2/q^2) [43]. Indeed, as has been demonstrated in [44, 45, 46, 47] small mass expansions can be a very effective tool for accurate predictions of mass dependences provided one is not too close to the threshold ($q^2 = m^2$ in our case). The exact two-loop result for the quark propagator can provide some insight into the accuracy of such an expansion.

We have analytically computed three terms in the small mass expansion of the quark propagator to order α_s^3. The calculation has been done with intensive use of computer algebra programs. In particular, we have used `QGRAF` [48] for the generation of diagrams and `LMP` [49] for the diagrammatic small mass expansion. The small mass expansion results in products of massless propagators and massive tadpoles. These have been evaluated with the help of the FORM packages MATAD [51] and `MINCER` [50] (A detailed description of the status of these algebraic programs can be found in [52]). It is convenient to write the functions $\Sigma_{S/V}$ in the following way:

$$\Sigma_{S/V} = 1 + \sum_{i\geq 1} \Sigma_{S/V}^{(i)}\; a_s^i(\mu) . \tag{28}$$

Our results for the separate contributions in the Landau gauge and in the $\overline{\text{MS}}$ scheme read [3] ($l_{qm} \equiv \ln(-q^2/\bar{m}^2)$, $l_{q\mu} \equiv \ln(-q^2/\mu^2)$, $z = m^2/q^2$, also note that in the Landau gauge $\Sigma_V^{(1)} \equiv 0$):

$$\begin{aligned}
\Sigma_V^{(2)} = {} & \left[\frac{359}{144} - \frac{7}{48}n_f - \frac{3}{4}\zeta_3 - \frac{67}{48}l_{q\mu} + \frac{1}{12}n_f l_{q\mu}\right] \\
& + z\left[\frac{79}{24} - \frac{1}{6}n_f - \frac{9}{4}\zeta_3 - l_{q\mu} - \frac{1}{2}l_{qm}\right] \\
& + z^2\left[-\frac{331}{216} - \frac{1}{24}n_f + \frac{10}{3}\zeta_3 - \frac{209}{72}l_{q\mu} - \frac{1}{6}n_f l_{q\mu} - \frac{19}{12}l_{q\mu}^2 - \frac{209}{144}l_{qm}\right. \\
& \qquad \left. - \frac{1}{12}n_f l_{qm} - \frac{19}{12}l_{q\mu}l_{qm} - \frac{19}{48}l_{qm}^2\right] \\
& + z^3\left[\frac{123}{32} + \frac{1}{18}n_f + \frac{109}{54}l_{q\mu} + \frac{7}{36}l_{q\mu}^2 + \frac{109}{108}l_{qm} + \frac{7}{36}l_{q\mu}l_{qm} + \frac{7}{144}l_{qm}^2\right] \\
& + z^4\left[\frac{668909}{172800} + \frac{1}{48}n_f + \frac{5093}{2160}l_{q\mu} - \frac{317}{72}l_{q\mu}^2 + \frac{5093}{4320}l_{qm} - \frac{317}{72}l_{q\mu}l_{qm} - \frac{317}{288}l_{qm}^2\right]
\end{aligned}$$

[3] Below we keep only the n_f dependence; the expressions including the full dependence on the gauge parameter and the group theoretical factors C_A, C_F and T are given in Appendix A. Note also that the result for three loop massless quark propagator can be also found in the source code of the updated version of the FORM program MINCER [53].

$$
\begin{aligned}
+ \quad & z^5 \left[\frac{23155073}{2592000} + \frac{1}{90}\,n_f + \frac{249353}{10800} l_{q\mu} - \frac{25553}{1080} l_{q\mu}^2 + \frac{249353}{21600} l_{qm} \right. \\
& \left. - \frac{25553}{1080} l_{q\mu} l_{qm} - \frac{25553}{4320} l_{qm}^2 \right],
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\Sigma_V^{(3)} = \quad & \left[\frac{439543}{10368} - \frac{12361}{2592}\,n_f + \frac{785}{7776}\,n_f^2 - \frac{8009}{384}\,\zeta_3 + \frac{55}{72}\,n_f\,\zeta_3 - \frac{79}{256}\,\zeta_4 + \frac{1165}{192}\,\zeta_5 \right. \\
& - \frac{52321}{2304} l_{q\mu} + \frac{559}{216}\,n_f l_{q\mu} - \frac{13}{216}\,n_f^2 l_{q\mu} + \frac{607}{128}\,\zeta_3 l_{q\mu} - \frac{1}{4}\,n_f\,\zeta_3 l_{q\mu} + \frac{737}{192} l_{q\mu}^2 \\
& \left. - \frac{133}{288}\,n_f l_{q\mu}^2 + \frac{1}{72}\,n_f^2 l_{q\mu}^2 \right] \\
+ \quad & z \left[\frac{5461}{96} - \frac{667}{144}\,n_f + \frac{5}{54}\,n_f^2 - \frac{10025}{288}\,\zeta_3 + \frac{65}{36}\,n_f\,\zeta_3 + \frac{3}{2}\,\zeta_4 - \frac{6235}{576}\,\zeta_5 - \frac{857}{24} l_{q\mu} \right. \\
& + \frac{185}{72}\,n_f l_{q\mu} - \frac{1}{18}\,n_f^2 l_{q\mu} + \frac{123}{8}\,\zeta_3 l_{q\mu} - \frac{3}{4}\,n_f\,\zeta_3 l_{q\mu} + \frac{9}{16} l_{q\mu}^2 - \frac{481}{96} l_{qm} + \frac{1}{9}\,n_f l_{qm} \\
& \left. - \frac{3}{4}\,\zeta_3 l_{qm} - \frac{51}{16} l_{q\mu} l_{qm} + \frac{1}{6}\,n_f l_{q\mu} l_{qm} - \frac{111}{64} l_{qm}^2 + \frac{1}{12}\,n_f l_{qm}^2 \right] \\
+ \quad & z^2 \left[-\frac{389057}{82944} - \frac{397}{1296}\,n_f + \frac{11}{216}\,n_f^2 + \frac{196475}{3456}\,\zeta_3 - \frac{719}{432}\,n_f\,\zeta_3 + \frac{35}{36}\,\zeta_4 - \frac{8815}{432}\,\zeta_5 \right. \\
& + \frac{19}{216}\,B_4 - \frac{8303}{384} l_{q\mu} - \frac{1697}{648}\,n_f l_{q\mu} + \frac{17}{216}\,n_f^2 l_{q\mu} - \frac{5177}{144}\,\zeta_3 l_{q\mu} + \frac{47}{18}\,n_f\,\zeta_3 l_{q\mu} \\
& - \frac{4739}{288} l_{q\mu}^2 + \frac{553}{432}\,n_f l_{q\mu}^2 - \frac{1}{18}\,n_f^2 l_{q\mu}^2 + \frac{973}{288} l_{q\mu}^3 - \frac{23}{54}\,n_f l_{q\mu}^3 - \frac{115007}{6912} l_{qm} \\
& - \frac{3029}{2592}\,n_f l_{qm} + \frac{5}{108}\,n_f^2 l_{qm} - \frac{617}{288}\,\zeta_3 l_{qm} + \frac{3}{4}\,n_f\,\zeta_3 l_{qm} - \frac{4127}{144} l_{q\mu} l_{qm} \\
& + \frac{35}{36}\,n_f l_{q\mu} l_{qm} - \frac{1}{36}\,n_f^2 l_{q\mu} l_{qm} - \frac{157}{64} l_{q\mu}^2 l_{qm} - \frac{3}{8}\,n_f l_{q\mu}^2 l_{qm} - \frac{3923}{384} l_{qm}^2 \\
& \left. + \frac{287}{1728}\,n_f l_{qm}^2 - \frac{1915}{384} l_{q\mu} l_{qm}^2 - \frac{1}{18}\,n_f l_{q\mu} l_{qm}^2 - \frac{3359}{2304} l_{qm}^3 + \frac{11}{864}\,n_f l_{qm}^3 \right],
\end{aligned} \tag{30}
$$

$$
\Sigma_S^{(1)} = \left[-\frac{4}{3} + l_{q\mu} - z - 2 l_{q\mu} z - l_{qm} z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \frac{1}{12} z^4 + \frac{1}{20} z^5 \right], \tag{31}
$$

$$
\begin{aligned}
\Sigma_S^{(2)} = \quad & \left[-\frac{5009}{288} + \frac{13}{18}\,n_f + \frac{47}{12}\,\zeta_3 + \frac{509}{48} l_{q\mu} - \frac{4}{9}\,n_f l_{q\mu} - \frac{15}{8} l_{q\mu}^2 + \frac{1}{12}\,n_f l_{q\mu}^2 \right] \\
+ \quad & z \left[-\frac{791}{144} + \frac{5}{18}\,n_f - \frac{43}{12}\,\zeta_3 - \frac{319}{24} l_{q\mu} + \frac{7}{18}\,n_f l_{q\mu} + \frac{7}{2} l_{q\mu}^2 - \frac{1}{3}\,n_f l_{q\mu}^2 \right. \\
& \left. - \frac{409}{48} l_{qm} + \frac{5}{18}\,n_f l_{qm} - \frac{9}{4} l_{q\mu} l_{qm} - \frac{1}{6}\,n_f l_{q\mu} l_{qm} - 2 l_{qm}^2 \right] \\
+ \quad & z^2 \left[\frac{13}{12} - \frac{7}{72}\,n_f - \frac{95}{24} l_{q\mu} + \frac{1}{4}\,n_f l_{q\mu} - \frac{49}{12} l_{q\mu}^2 - \frac{1}{24} l_{qm} + \frac{1}{12}\,n_f l_{qm} \right. \\
& \left. - \frac{49}{12} l_{q\mu} l_{qm} - \frac{49}{48} l_{qm}^2 \right] \\
+ \quad & z^3 \left[-\frac{451}{324} - \frac{5}{54}\,n_f + \frac{829}{216} l_{q\mu} + \frac{1}{12}\,n_f l_{q\mu} + \frac{527}{108} l_{q\mu}^2 + \frac{295}{108} l_{qm} \right. \\
& \left. + \frac{1}{36}\,n_f l_{qm} + \frac{527}{108} l_{q\mu} l_{qm} + \frac{527}{432} l_{qm}^2 \right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad z^4 \left[-\frac{211333}{20736} - \frac{53}{864}\,n_f - \frac{4831}{432}l_{q\mu} + \frac{1}{24}\,n_f l_{q\mu} + \frac{4843}{216}l_{q\mu}^2 - \frac{551}{108}l_{qm} \right.\\
&\qquad \left. + \frac{1}{72}\,n_f l_{qm} + \frac{4843}{216}l_{q\mu}l_{qm} + \frac{4843}{864}l_{qm}^2\right]\\
&+ \quad z^5 \left[-\frac{24966073}{864000} - \frac{77}{1800}\,n_f - \frac{349147}{3600}l_{q\mu} + \frac{1}{40}\,n_f l_{q\mu} + \frac{3539}{40}l_{q\mu}^2 \right.\\
&\qquad \left. - \frac{21667}{450}l_{qm} + \frac{1}{120}\,n_f l_{qm} + \frac{3539}{40}l_{q\mu}l_{qm} + \frac{3539}{160}l_{qm}^2\right], \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_S^{(3)} = &\left[-\frac{1612847}{6912} + \frac{79621}{3888}\,n_f - \frac{191}{729}\,n_f^2 + \frac{150265}{1728}\,\zeta_3 - \frac{755}{216}\,n_f\,\zeta_3 - \frac{1}{54}\,n_f^2\,\zeta_3 \right.\\
&+ \frac{79}{256}\,\zeta_4 + \frac{5}{12}\,n_f\,\zeta_4 - \frac{6455}{576}\,\zeta_5 + \frac{40329}{256}l_{q\mu} - \frac{6083}{432}\,n_f l_{q\mu} + \frac{73}{324}\,n_f^2 l_{q\mu}\\
&- \frac{10013}{384}\,\zeta_3 l_{q\mu} + \frac{17}{36}\,n_f\,\zeta_3 l_{q\mu} - \frac{2589}{64}l_{q\mu}^2 + \frac{119}{32}\,n_f l_{q\mu}^2 - \frac{2}{27}\,n_f^2 l_{q\mu}^2 + \frac{65}{16}l_{q\mu}^3\\
&\left. - \frac{7}{18}\,n_f l_{q\mu}^3 + \frac{1}{108}\,n_f^2 l_{q\mu}^3\right]\\
+ \quad z &\left[-\frac{303803}{6912} + \frac{2969}{648}\,n_f - \frac{25}{324}\,n_f^2 - \frac{178745}{3456}\,\zeta_3 + \frac{313}{54}\,n_f\,\zeta_3 - \frac{1}{9}\,n_f^2\,\zeta_3 - \frac{13}{3}\,\zeta_4 \right.\\
&- \frac{9635}{1728}\,\zeta_5 + \frac{2}{9}\,B_4 - \frac{172685}{1152}l_{q\mu} + \frac{4703}{432}\,n_f l_{q\mu} - \frac{5}{81}\,n_f^2 l_{q\mu} + \frac{7493}{192}\,\zeta_3 l_{q\mu}\\
&+ \frac{17}{36}\,n_f\,\zeta_3 l_{q\mu} + \frac{367}{6}l_{q\mu}^2 - \frac{193}{24}\,n_f l_{q\mu}^2 + \frac{17}{108}\,n_f^2 l_{q\mu}^2 - \frac{135}{8}l_{q\mu}^3 + \frac{14}{9}\,n_f l_{q\mu}^3\\
&- \frac{1}{18}\,n_f^2 l_{q\mu}^3 - \frac{73547}{768}l_{qm} + \frac{839}{108}\,n_f l_{qm} - \frac{25}{324}\,n_f^2 l_{qm} + \frac{1645}{384}\,\zeta_3 l_{qm}\\
&+ \frac{5}{6}\,n_f\,\zeta_3 l_{qm} - \frac{431}{32}l_{q\mu}l_{qm} - \frac{391}{144}\,n_f l_{q\mu}l_{qm} + \frac{5}{54}\,n_f^2 l_{q\mu}l_{qm} - \frac{167}{16}l_{q\mu}^2 l_{qm}\\
&+ \frac{1}{3}\,n_f l_{q\mu}^2 l_{qm} - \frac{1}{36}\,n_f^2 l_{q\mu}^2 l_{qm} - \frac{2359}{96}l_{qm}^2 + \frac{17}{18}\,n_f l_{qm}^2 - 10 l_{q\mu}l_{qm}^2\\
&\left. - \frac{9}{2}l_{qm}^3 + \frac{1}{9}\,n_f l_{qm}^3\right]\\
+ \quad z^2 &\left[\frac{882619}{82944} - \frac{3151}{1728}\,n_f - \frac{35}{648}\,n_f^2 - \frac{34153}{1728}\,\zeta_3 + \frac{91}{144}\,n_f\,\zeta_3 - 3\,\zeta_4 - \frac{55}{54}\,\zeta_5 \right.\\
&- \frac{131681}{6912}l_{q\mu} + \frac{3293}{432}\,n_f l_{q\mu} - \frac{13}{72}\,n_f^2 l_{q\mu} - \frac{3251}{576}\,\zeta_3 l_{q\mu} - \frac{1203}{32}l_{q\mu}^2\\
&+ \frac{35}{432}\,n_f l_{q\mu}^2 + \frac{5}{72}\,n_f^2 l_{q\mu}^2 + \frac{5401}{432}l_{q\mu}^3 - \frac{49}{54}\,n_f l_{q\mu}^3 + \frac{47095}{13824}l_{qm}\\
&+ \frac{2285}{864}\,n_f l_{qm} - \frac{2}{27}\,n_f^2 l_{qm} - \frac{3251}{1152}\,\zeta_3 l_{qm} - \frac{1303}{24}l_{q\mu}l_{qm} + \frac{887}{432}\,n_f l_{q\mu}l_{qm}\\
&+ \frac{1}{36}\,n_f^2 l_{q\mu}l_{qm} - \frac{773}{288}l_{q\mu}^2 l_{qm} - \frac{49}{72}\,n_f l_{q\mu}^2 l_{qm} - \frac{2431}{192}l_{qm}^2 + \frac{1271}{1728}\,n_f l_{qm}^2\\
&\left. - \frac{6947}{576}l_{q\mu}l_{qm}^2 - \frac{13121}{3456}l_{qm}^3 + \frac{49}{864}\,n_f l_{qm}^3\right]. \qquad (33)
\end{aligned}
$$

4 RI scheme versus $\overline{\rm MS}$ and MOM schemes

4.1 Three Loop Conversion functions

A direct use of Eqs. (19, 20, 22) and (23) leads to the following analytical expressions for the conversion factors between the $\overline{\rm MS}$ and RI schemes. The results are shown for QCD ($SU(3)$) and Landau gauge as functions of n_f (the ζ_i are the values $\zeta(i)$ of Riemann's Zeta function):

$$\begin{aligned} C_2^{\rm RI} = \quad & 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{517}{18} + 12\,\zeta_3 + \frac{5}{3}\,n_f\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{1287283}{648} + \frac{14197}{12}\,\zeta_3 + \frac{79}{4}\,\zeta_4 - \frac{1165}{3}\,\zeta_5 \right. \\ & \left. + \frac{18014}{81}\,n_f - \frac{368}{9}\,\zeta_3\,n_f - \frac{1102}{243}\,n_f^2\right], \end{aligned} \tag{34}$$

$$\begin{aligned} C_m^{\rm RI} = \quad & 1 + \frac{\alpha_s}{4\pi}\left[-\frac{16}{3}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{1990}{9} + \frac{152}{3}\,\zeta_3 + \frac{89}{9}\,n_f\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{6663911}{648} + \frac{408007}{108}\,\zeta_3 - \frac{2960}{9}\,\zeta_5 + \frac{236650}{243}\,n_f \right. \\ & \left. - \frac{4936}{27}\,\zeta_3\,n_f + \frac{80}{3}\,\zeta_4\,n_f - \frac{8918}{729}\,n_f^2 - \frac{32}{27}\,\zeta_3\,n_f^2\right], \end{aligned} \tag{35}$$

$$\begin{aligned} C_2^{\rm RI'} = \quad & 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{359}{9} + 12\,\zeta_3 + \frac{7}{3}\,n_f\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{439543}{162} + \frac{8009}{6}\,\zeta_3 + \frac{79}{4}\,\zeta_4 - \frac{1165}{3}\,\zeta_5 \right. \\ & \left. + \frac{24722}{81}\,n_f - \frac{440}{9}\,\zeta_3\,n_f - \frac{1570}{243}\,n_f^2\right], \end{aligned} \tag{36}$$

$$\begin{aligned} C_m^{\rm RI'} = \quad & 1 + \frac{\alpha_s}{4\pi}\left[-\frac{16}{3}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{3779}{18} + \frac{152}{3}\,\zeta_3 + \frac{83}{9}\,n_f\right] \\ & + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{3115807}{324} + \frac{195809}{54}\,\zeta_3 - \frac{2960}{9}\,\zeta_5 + \frac{217390}{243}\,n_f \right. \\ & \left. - \frac{4720}{27}\,\zeta_3\,n_f + \frac{80}{3}\,\zeta_4\,n_f - \frac{7514}{729}\,n_f^2 - \frac{32}{27}\,\zeta_3\,n_f^2\right]. \end{aligned} \tag{37}$$

At a scale $\mu = 2$ GeV and $n_f = 4$, the numerical contributions of the leading order to NNNLO terms are as follows (for simplicity we inserted $\alpha_s/\pi = 0.1$):

$$C_2^{\rm RI} = 1.0 + 0.0 - 0.00476 - 0.00508\,, \tag{38}$$

$$C_m^{\rm RI} = 1. - 0.1333 - 0.0754 - 0.0495 \tag{39}$$

and

$$C_2^{\rm RI'} = 1.0 + 0.0 - 0.0101 - 0.0095\,, \tag{40}$$

$$C_m^{\rm RI'} = 1.0 - 0.1333 - 0.0701 - 0.0458\,. \tag{41}$$

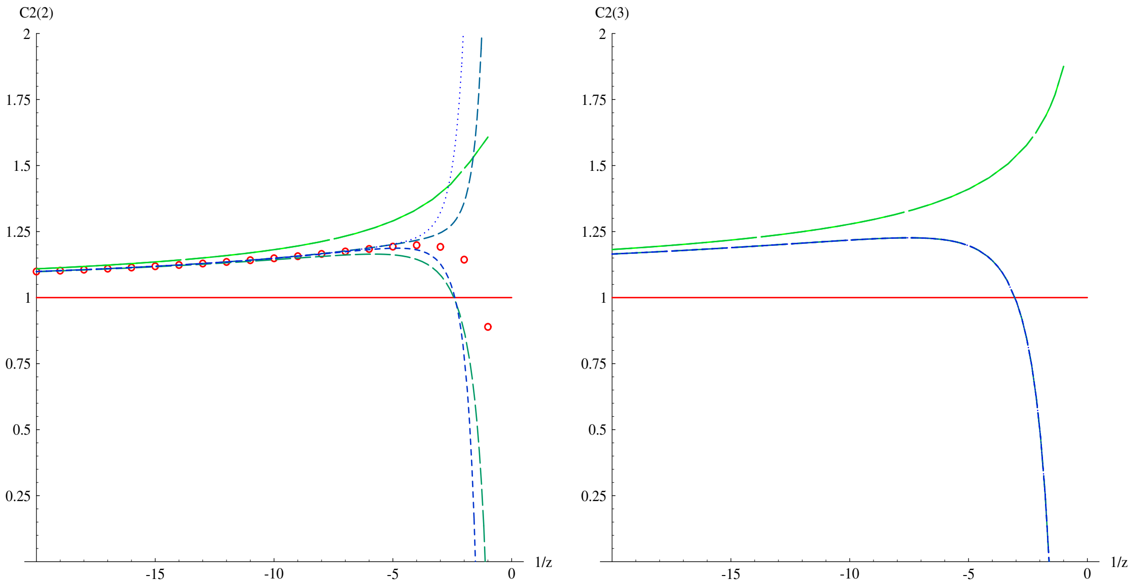

Figure 1: The ratio of the RI and MOM scheme conversion functions $C_{2,l}^{MOM}/C_{2,l}^{\mathrm{RI}}$ as functions of $-\mu^2/m^2$. Shown are the coefficients in the expansion in a_s/π as expansions to order z to z^5 (z^2 for 3 loops). Note that in Landau gauge $C_2^? = 0$, for 2 loops some numeric values for the exact mass dependence are shown as well.

One observes that the sizes of the NNLO and NNNLO contributions to C_m^{RI} at this scale amount to about 7.5% and 5% respectively. This shows that perturbation theory can not be used for a precise conversion of the RI quark masses to the $\overline{\mathrm{MS}}$ ones at the renormalization scale $\mu = 2$ GeV. The convergence can be improved if one increases μ to, say, 3 GeV. Indeed, with this choice of μ the standard three-loop evolution gives $\alpha_s(3\text{ GeV}) = 0.262$ and Eqs. (38,39) transform to

$$C_2^{RI} = 1.0 + 0.0 - 0.00333 - 0.00296 \tag{42}$$

and

$$C_m^{RI} = 1.0 - 0.111 - 0.0526 - 0.0289. \tag{43}$$

The accuracy of the massless approximation can be tested by computing the ratio $C_?^{\mathrm{RI}}/C_?^{MOM}$ as a series in $z = m_q^2/(-\mu^2)$. Using the results of the previous section one obtains

$$\begin{aligned}
\frac{C_m^{\mathrm{RI}}}{C_m^{MOM}} = \quad & 1. \\
+ \quad & a_s \left[-1.z - 0.5z^2 + 0.16667z^3 - 0.083333z^4 + 0.05z^5 - 1.zl_z\right] \\
+ \quad & a_s^2 \left[-7.6458z + 0.86458z^2 - 3.5129z^3 + 13.913z^4 - 36.828z^5 \right. \\
& \quad - 6.3264zl_z + 0.10417z^2l_z + 2.3866z^3l_z + 7.3259z^4l_z - 71.347z^5l_z \\
& \quad \left. - 2.zl_z^2 + 1.0208z^2l_z^2 + 1.1956z^3l_z^2 - 6.706z^4l_z^2 + 30.991z^5l_z^2\right] \\
+ \quad & a_s^3 \left[-59.008z + 48.743z^2 - 41.464zl_z - 5.4235z^2l_z - 18.829zl_z^2\right.
\end{aligned}$$

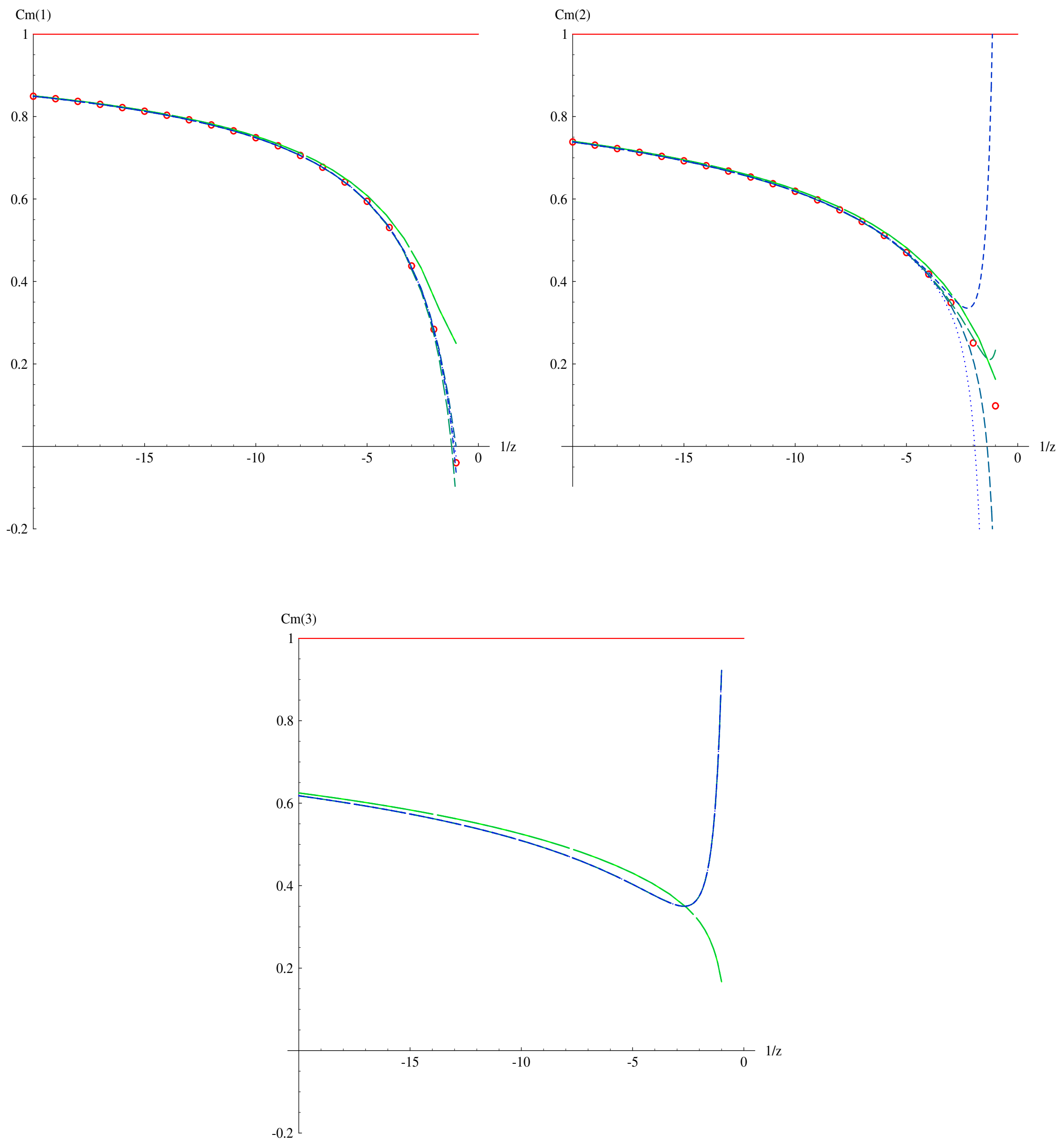

Figure 2: The ratio of the RI and MOM scheme conversion functions $C^{MOM}_{m,l}/C^{\mathrm{RI}}_{m,l}$ as functions of $-\mu^2/m^2$. Shown are the coefficients in the expansion in a_s/π as expansions to order z to z^5 (z^2 for 3 loops). For 1 and 2 loops some numeric values for the exact mass dependence are shown as well.

$$+ 9.1024z^2 l_z^2 - 4.0556 z l_z^3 + 3.5697 z^2 l_z^3 \big] \,, \tag{44}$$

$$\begin{aligned}
\frac{C_m^{\rm RI}}{C_m^{MOM}} = \quad & 1. \\
+ \quad & a_s^2 \big[0.28981z - 0.89236z^2 + 1.5284z^3 - 3.3649z^4 + 7.6945z^5 \\
& \quad + 0.25 z l_z - 0.39583 z^2 l_z + 0.45602 z^3 l_z - 2.2796 z^4 l_z + 23.231 z^5 l_z \\
& \quad + 0.024306 z^3 l_z^2 + 1.1007 z^4 l_z^2 - 8.8726 z^5 l_z^2 \big] \\
+ \quad & a_s^3 \big[4.4496z - 20.979z^2 + 2.5931 z l_z - 3.0558 z^2 l_z + 0.70052 z l_z^2 \\
& \quad - 0.49414 z^2 l_z^2 \big] \,,
\end{aligned} \tag{45}$$

where $l_z = log(-m^2/\mu^2)$ and we have evaluated the coefficients in the series in z with $n_f = 4$.

To illustrate the quality of these expansions, we have plotted (see figures 1 and 2) the ratio of the 1, 2 and 3 loop coefficients of C_m^{MOM} and $C_m^{\rm RI}$ as functions of $1/z = -\mu^2/m^2$ in the Landau gauge and for simplicity with $n_f = 4$ for all values of z. The circles in the plots for 1 and 2 loops correspond to the exact results from [41]. The convergence of the small mass (corresponding to large negative values of $1/z$) expansions is good for $1/z < -4$, where the expansions for higher orders of z are almost indistinguishable as well among each other as well as from the numbers received from the exact 2 loop propagator. On the other hand, due to the $z \ln(z)^i, i = 1, \ldots, l$ (where l is the number of loops) terms, the MOM coefficients are approaching the corresponding values in the RI -scheme for increasing $|z|$ only very slowly. This makes the RI -scheme as an approximation to the MOM-scheme for the c quark useless.

4.2 Four Loop Quark Anomalous Dimensions

We start from the $\overline{\rm MS}$ scheme. The quark mass anomalous dimension was computed at four loops quite recently in [54, 55] and reads

$$\gamma_m(a_s) \equiv - \sum_{i\geq 0} \gamma_m^{(i)} a_s^{i+1} \,,$$

$$\gamma_m^{(0)} = \quad 1, \tag{46}$$

$$\gamma_m^{(1)} = \frac{1}{16} \left\{ \quad \frac{202}{3} \quad + \quad n_f \left[-\frac{20}{9} \right] \quad \right\}, \tag{47}$$

$$\gamma_m^{(2)} = \frac{1}{64} \left\{ \quad 1249 + n_f \left[-\frac{2216}{27} - \frac{160}{3} \zeta_3 \right] + n_f^2 \left[-\frac{140}{81} \right] \quad \right\}, \tag{48}$$

$$\begin{aligned}
\gamma_m^{(3)} = \frac{1}{256} \Bigg\{ \quad & \left[\frac{4603055}{162} + \frac{135680}{27} \zeta_3 - 8800 \zeta_5 \right] \\
+ \quad & n_f \left[-\frac{91723}{27} - \frac{34192}{9} \zeta_3 + 880 \zeta_4 + \frac{18400}{9} \zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{5242}{243} + \frac{800}{9} \zeta_3 - \frac{160}{3} \zeta_4 \right] \quad + \quad n_f^3 \left[-\frac{332}{243} + \frac{64}{27} \zeta_3 \right] \quad \Bigg\}.
\end{aligned} \tag{49}$$

The result for the quark field anomalous dimension was found by one of the authors in the course of computing γ_m [54] and read for the QCD case in Landau gauge[4]

$$\gamma_2(a_s) \equiv -\sum_{i\geq 0} \gamma_2^{(i)} a_s^{i+1},$$

$$\gamma_2^{(1)} = \frac{1}{16}\left\{ \frac{67}{3} + n_f\left[-\frac{4}{3}\right] \right\}, \tag{50}$$

$$\gamma_2^{(2)} = \frac{1}{64}\left\{ \left[\frac{20729}{36} - \frac{79}{2}\zeta_3\right] + n_f\left[-\frac{550}{9}\right] + n_f^2\left[\frac{20}{27}\right] \right\}, \tag{51}$$

$$\begin{aligned}\gamma_2^{(3)} = \frac{1}{256}\Bigg\{ & \left[\frac{2109389}{162} - \frac{565939}{324}\zeta_3 + \frac{2607}{4}\zeta_4 - \frac{761525}{1296}\zeta_5\right] \\ + & n_f\left[-\frac{162103}{81} - \frac{2291}{27}\zeta_3 - \frac{79}{2}\zeta_4 - \frac{160}{3}\zeta_5\right] \\ + & n_f^2\left[\frac{3853}{81} + \frac{160}{9}\zeta_3\right] + n_f^3\left[\frac{140}{243}\right] \Bigg\}.\end{aligned} \tag{52}$$

For completeness we also give the field anomalous dimension for the case of QED with n_f different fermion species[5]:

$$\gamma_2^{QED\,(0)} = \frac{1}{4}\left[\xi_L\right], \tag{53}$$

$$\gamma_2^{QED\,(1)} = \frac{1}{16}\left\{ -\frac{3}{2} + n_f\left[-2\right] \right\}, \tag{54}$$

$$\gamma_2^{QED\,(2)} = \frac{1}{64}\left\{ \frac{3}{2} + n_f\left[3\right] + n_f^2\left[\frac{20}{9}\right] \right\}, \tag{55}$$

$$\begin{aligned}\gamma_2^{QED\,(3)} = \frac{1}{256}\Bigg\{ & \left[-\frac{1027}{8} - 400\,\zeta_3 + 640\,\zeta_5\right] + n_f\left[\frac{460}{3} - 64\,\zeta_3\right] \\ + & n_f^2\left[\frac{304}{9} - 32\,\zeta_3\right] + n_f^3\left[\frac{280}{81}\right] \Bigg\}.\end{aligned} \tag{56}$$

In order to compute the corresponding anomalous dimensions for the RI and RI' schemes, one just needs to make use of Eqs. (46) to (49) in combination with the three loop conversion functions from section 4.1. As a result for the QCD case we get for the RI scheme:

[4] γ_2 is gauge dependent and in the Landau gauge $\gamma_2^{(0)} = 0$; the results for SU(N) group in general covariant gauge are given in Appendix C.

[5] For the QED case, all gauge dependence is in $\gamma_2^{(0)}$, where ξ_L is defined in Appendix A ($\xi_L = 0$ for Landau gauge)

$$\gamma_m^{\mathrm{RI}\,(0)} = 1, \tag{57}$$

$$\gamma_m^{\mathrm{RI}\,(1)} = \frac{1}{16}\left\{ 126 + n_f\left[-\frac{52}{9}\right] \right\}, \tag{58}$$

$$\gamma_m^{\mathrm{RI}\,(2)} = \frac{1}{64}\left\{ \left[\frac{20911}{3} - \frac{3344}{3}\zeta_3\right] + n_f\left[-\frac{18386}{27} + \frac{128}{9}\zeta_3\right] + n_f^2\left[\frac{928}{81}\right] \right\}, \tag{59}$$

$$\begin{aligned}\gamma_m^{\mathrm{RI}\,(3)} = \frac{1}{256}\Bigg\{ & \left[\frac{300665987}{648} - \frac{15000871}{108}\zeta_3 + \frac{6160}{3}\zeta_5\right] \\ + \; & n_f\left[-\frac{7535473}{108} + \frac{627127}{54}\zeta_3 + \frac{4160}{3}\zeta_5\right] \\ + \; & n_f^2\left[\frac{670948}{243} - \frac{6416}{27}\zeta_3\right] + n_f^3\left[-\frac{18832}{729}\right] \Bigg\}.\end{aligned} \tag{60}$$

The corresponding equations for the RI' scheme are:

$$\gamma_m^{\mathrm{RI'}(0)} = 1, \tag{61}$$

$$\gamma_m^{\mathrm{RI'}(1)} = \frac{1}{16}\left\{ 126 + n_f\left[-\frac{52}{9}\right] \right\}, \tag{62}$$

$$\gamma_m^{\mathrm{RI'}(2)} = \frac{1}{64}\left\{ \left[\frac{20174}{3} - \frac{3344}{3}\zeta_3\right] + n_f\left[-\frac{17588}{27} + \frac{128}{9}\zeta_3\right] + n_f^2\left[\frac{856}{81}\right] \right\}, \tag{63}$$

$$\begin{aligned}\gamma_m^{\mathrm{RI'}(3)} = \frac{1}{256}\Bigg\{ & \left[\frac{141825253}{324} - \frac{7230017}{54}\zeta_3 + \frac{6160}{3}\zeta_5\right] \\ + \; & n_f\left[-\frac{3519059}{54} + \frac{298241}{27}\zeta_3 + \frac{4160}{3}\zeta_5\right] \\ + \; & n_f^2\left[\frac{611152}{243} - \frac{5984}{27}\zeta_3\right] + n_f^3\left[-\frac{16024}{729}\right] \Bigg\}.\end{aligned} \tag{64}$$

The quark field anomalous dimensions for the RI and RI' schemes are given in Appendix C.

4.3 NNNLO relation for the RI quark mass and the RGI mass $\hat{m}_q$

It is customary to solve the RG equation (6) for the quark running mass $m(\mu)$ as follows

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \tag{65}$$

where (x stands for either $\alpha_s(\mu)/\pi$ or $\alpha_s(\mu_0)/\pi$)

$$
\begin{aligned}
c(x) &= \exp\left\{\int^x \mathrm{d}x' \frac{\gamma_m(x')}{\beta(x')}\right\} \\
&= (x)^{\bar\gamma_0}\left\{1 + (\bar\gamma_1 - \bar\beta_1\bar\gamma_0)x\right. \\
&+ \frac{1}{2}\left[(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)^2 + \bar\gamma_2 + \bar\beta_1{}^2\bar\gamma_0 - \bar\beta_1\bar\gamma_1 - \bar\beta_2\bar\gamma_0\right]x^2 \\
&+ \left[\frac{1}{6}(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)^3 + \frac{1}{2}(\bar\gamma_1 - \bar\beta_1\bar\gamma_0)(\bar\gamma_2 + \bar\beta_1{}^2\bar\gamma_0 - \bar\beta_1\bar\gamma_1 - \bar\beta_2\bar\gamma_0)\right. \\
&\left.\left. + \frac{1}{3}\left(\bar\gamma_3 - \bar\beta_1{}^3\bar\gamma_0 + 2\bar\beta_1\bar\beta_2\bar\gamma_0 - \bar\beta_3\bar\gamma_0 + \bar\beta_1{}^2\bar\gamma_1 - \bar\beta_2\bar\gamma_1 - \bar\beta_1\bar\gamma_2\right)\right]x^3 + \mathcal{O}(x^4)\right\}.
\end{aligned}
\tag{66}
$$

Here $\bar\gamma_i = \gamma_m^{(i)}/\beta_0$, $\bar\beta_i = \beta_i/\beta_0$, ($i$=1,2,3) and β_i are the coefficients of the QCD beta-function as defined in Eq. ((5)). The four-loop beta-function has recently been computed in [56] with the result

$$
\begin{aligned}
\beta_0 &= \frac{1}{4}\left(11 - \frac{2}{3}n_f\right), \quad \beta_1 = \frac{1}{16}\left(102 - \frac{38}{3}n_f\right), \\
\beta_2 &= \frac{1}{64}\left(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right), \\
\beta_3 &= \frac{1}{256}\left(\frac{149753}{6} + 3564\zeta(3)\right. \\
&\left. - \left[\frac{1078361}{162} + \frac{6508}{27}\zeta(3)\right]n_f + \left[\frac{50065}{162} + \frac{6472}{81}\zeta(3)\right]n_f^2 + \frac{1093}{729}n_f^3\right).
\end{aligned}
\tag{67}
$$

Eq. (65) directly leads to the RG invariant μ independent mass $\hat m_q$

$$
\hat m_q = \frac{m_q(\mu)}{c(a_s(\mu))}. \tag{68}
$$

An important property of $\hat m_q$ is its μ and scheme independence. The latter follows from the fact that $\hat m_q$ could be alternatively defined as follows

$$
\hat m_q = \lim_{\mu\to\infty} m_q(\mu)\left(\frac{\alpha_s(\mu)}{\pi}\right)^{-\frac{\gamma_m^0}{\beta_0}} \tag{69}
$$

and from the well-known universality of the one loop coefficients of the quark mass anomalous dimension and the β-function.

Evaluating the four loop approximation of the c-function in the RI and RI' schemes, we can state our results for the conversion functions as a relation between the RG invariant mass $\hat m$ and the masses m^{RI} and $m^{\mathrm{RI'}}$:

For $n_f = 3, 4, 5$ Eq. (68) assumes the form

$$
\begin{aligned}
\frac{\hat m^{(3)}}{m^{\mathrm{RI}}} = \left(\frac{\alpha_s}{\pi}\right)^{-\frac{4}{9}}\Bigg\{ & 1 + \frac{\alpha_s}{4\pi}\left[-\frac{722}{81}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{2521517}{13122} + \frac{536}{9}\zeta_3\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3\left[-\frac{88484924345}{12754584} + \frac{3089567}{972}\zeta_3 - \frac{18640}{81}\zeta_5\right]\Bigg\},
\end{aligned}
\tag{70}
$$

$$
\begin{aligned}
\frac{\hat m^{(4)}}{m^{\mathrm{RI}}} = \left(\frac{\alpha_s}{\pi}\right)^{-\frac{12}{25}}\Bigg\{ & 1 + \frac{\alpha_s}{4\pi}\left[-\frac{17606}{1875}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{3819632767}{21093750} + \frac{952}{15}\zeta_3\right] \\
& + \left(\frac{\alpha_s}{4\pi}\right)^3\left[-\frac{8512503162869851}{1423828125000} + \frac{1035345331}{337500}\zeta_3 - 304\,\zeta_5\right]\Bigg\},
\end{aligned}
\tag{71}
$$

$$\frac{\hat{m}^{(5)}}{m^{\rm RI}} = \left(\frac{\alpha_s}{\pi}\right)^{-\frac{12}{23}} \left\{ 1 + \frac{\alpha_s}{4\pi}\left[-\frac{15926}{1587}\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{2559841211}{15111414} + \frac{4696}{69}\zeta_3\right] + \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{4334826270205387}{863345304648} + \frac{3889063057}{1314036}\zeta_3 - \frac{26960}{69}\zeta_5\right] \right\}. \quad (72)$$

5 Conclusions

In this paper we have analytically computed the first few terms of the high-energy expansion of the three-loop quark propagator. These results have been used to find the NNNLO conversion factors transforming the $\overline{\rm MS}$ quark mass and the renormalized quark field to those defined in the RI scheme which is more suitable for lattice QCD calculations. The newly computed NNNLO corrections are numerically significant and should be taken into account when transforming the RI quark masses to the $\overline{\rm MS}$ ones.

We also have presented the four loop results for the quark mass and wave function anomalous dimensions in the RI and RI' schemes.

In principle, the knowledge of $\rm N^4LO$ conversion factors would be useful to even better control the convergence of the perturbation series. Unfortunately, such a calculation requires the knowledge of the quark propagator at four loops — a problem certainly out of the range of present calculational techniques.

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Note added

The main new findings of the present work have been communicated before publication to David Broadhurst (the four loop quark wave function anomalous dimension in the $\overline{\rm MS}$ scheme) and to Damir Becirevic (the NNNLO transformation formulae as well as the four loop quark mass anomalous dimensions in RI and RI' schemes). As a result, in [57] the NNNLO $\overline{\rm MS}$ — RI conversion relation have been used to transform the lattice results for the RI light quark masses into those for the $\overline{\rm MS}$ ones. In [58] the results for the (QED) fermion mass and field anomalous dimensions (Eqs. (54,55, 56) and Eqs. (7-9) of [54]) have been reproduced within an entirely different approach.

Appendixes

A Quark Propagators

Below we list the full three loop results for the quark propagator[6] computed in general covariant gauge with the tree gluon propagator

$$\frac{1}{q^2}(g_{\mu\nu} - (1-\xi_L)q_\mu q_\nu/q^2).$$

For $SU(N)$ gauge group colour factors have the values $C_A = N$, $C_F = (N^2-1)/(2N)$ and T $= 1/2$. The QED case is obtained with substitutions $C_A \to 0$, $C_F \to 1$ and $T \to 1$.

$$\Sigma_s^{(1)} = C_F \left[-1 - \frac{1}{2}\,\xi_L + \frac{3}{4} l_{q\mu} + \frac{1}{4}\,\xi_L l_{q\mu} \right], \tag{A.1}$$

$$\begin{aligned}
\Sigma_s^{(2)} = \quad & C_F^2 \left[-\frac{13}{16} - \frac{3}{4}\,\zeta_3 - \frac{1}{2}\,\xi_L - \frac{1}{16}\xi_L^2 + \frac{3}{4} l_{q\mu} + \frac{5}{8}\,\xi_L l_{q\mu} + \frac{1}{8}\xi_L^2 l_{q\mu} \right. \\
& \left. - \frac{9}{32} l_{q\mu}^2 - \frac{3}{16}\,\xi_L l_{q\mu}^2 - \frac{1}{32}\xi_L^2 l_{q\mu}^2 \right] \\
+ \quad & C_F\, C_A \left[-\frac{1531}{384} + \frac{21}{16}\,\zeta_3 - \frac{5}{8}\,\xi_L + \frac{3}{16}\,\zeta_3\,\xi_L - \frac{15}{128}\xi_L^2 + \frac{445}{192} l_{q\mu} \right. \\
& \left. + \frac{5}{16}\,\xi_L l_{q\mu} + \frac{5}{64}\xi_L^2 l_{q\mu} - \frac{11}{32} l_{q\mu}^2 - \frac{3}{64}\,\xi_L l_{q\mu}^2 - \frac{1}{64}\xi_L^2 l_{q\mu}^2 \right] \\
+ \quad & C_F\, T\, n_f \left[\frac{13}{12} - \frac{2}{3} l_{q\mu} + \frac{1}{8} l_{q\mu}^2 \right],
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Sigma_s^{(3)} = \quad & C_F^3 \left[-\frac{229}{48} - \frac{19}{32}\,\zeta_3 + \frac{15}{8}\,\zeta_5 - \frac{29}{64}\,\xi_L - \frac{21}{32}\,\zeta_3\,\xi_L - \frac{1}{16}\xi_L^2 + \frac{3}{32}\,\zeta_3 \xi_L^2 \right. \\
& + \frac{1}{96}\,\zeta_3 \xi_L^3 + \frac{105}{64} l_{q\mu} + \frac{9}{16}\,\zeta_3 l_{q\mu} + \frac{37}{64}\,\xi_L l_{q\mu} + \frac{3}{16}\,\zeta_3\,\xi_L l_{q\mu} \\
& + \frac{11}{64}\xi_L^2 l_{q\mu} + \frac{1}{64}\xi_L^3 l_{q\mu} - \frac{9}{32} l_{q\mu}^2 - \frac{21}{64}\,\xi_L l_{q\mu}^2 - \frac{1}{8}\xi_L^2 l_{q\mu}^2 - \frac{1}{64}\xi_L^3 l_{q\mu}^2 \\
& \left. + \frac{9}{128} l_{q\mu}^3 + \frac{9}{128}\,\xi_L l_{q\mu}^3 + \frac{3}{128}\xi_L^2 l_{q\mu}^3 + \frac{1}{384}\xi_L^3 l_{q\mu}^3 \right] \\
+ \quad & C_F^2\, C_A \left[-\frac{3005}{1152} - \frac{313}{96}\,\zeta_3 - \frac{3}{32}\,\zeta_4 + \frac{5}{8}\,\zeta_5 - \frac{1861}{768}\,\xi_L + \frac{43}{64}\,\zeta_3\,\xi_L - \frac{5}{16}\,\zeta_5\,\xi_L \right. \\
& - \frac{35}{128}\xi_L^2 - \frac{5}{256}\xi_L^3 - \frac{1}{64}\,\zeta_3 \xi_L^3 + \frac{2467}{512} l_{q\mu} + \frac{37}{64}\,\zeta_3 l_{q\mu} + \frac{4511}{1536}\,\xi_L l_{q\mu} \\
& - \frac{15}{32}\,\zeta_3\,\xi_L l_{q\mu} + \frac{221}{512}\xi_L^2 l_{q\mu} - \frac{3}{64}\,\zeta_3 \xi_L^2 l_{q\mu} + \frac{27}{512}\xi_L^3 l_{q\mu} - \frac{533}{256} l_{q\mu}^2 \\
& - \frac{793}{768}\,\xi_L l_{q\mu}^2 - \frac{45}{256}\xi_L^2 l_{q\mu}^2 - \frac{7}{256}\xi_L^3 l_{q\mu}^2 + \frac{33}{128} l_{q\mu}^3 + \frac{31}{256}\,\xi_L l_{q\mu}^3 \\
& \left. + \frac{3}{128}\xi_L^2 l_{q\mu}^3 + \frac{1}{256}\xi_L^3 l_{q\mu}^3 \right] \\
+ \quad & C_F^2\, T\, n_f \left[\frac{4699}{1152} - \frac{11}{12}\,\zeta_3 - \frac{3}{8}\,\zeta_4 + \frac{23}{48}\,\xi_L + \frac{1}{8}\,\zeta_3\,\xi_L - \frac{167}{64} l_{q\mu} + \frac{1}{4}\,\zeta_3 l_{q\mu} \right.
\end{aligned}$$

[6] for the exact definition of the Σ's see Eqs. (2) and (28)

$$
\begin{aligned}
& - \frac{29}{48}\,\xi_L l_{q\mu} + \frac{23}{32} l_{q\mu}^2 + \frac{11}{48}\,\xi_L l_{q\mu}^2 - \frac{3}{32} l_{q\mu}^3 - \frac{1}{32}\,\xi_L l_{q\mu}^3 \Bigg] \\
+ \quad & C_F C_A^2 \Bigg[-\frac{4315565}{248832} + \frac{20305}{2304}\,\zeta_3 + \frac{69}{1024}\,\zeta_4 - \frac{405}{256}\,\zeta_5 - \frac{66301}{36864}\,\xi_L + \frac{193}{256}\,\zeta_3\,\xi_L \\
& - \frac{3}{512}\,\zeta_4\,\xi_L - \frac{5}{128}\,\zeta_5\,\xi_L - \frac{1377}{4096}\xi_L^2 + \frac{3}{32}\,\zeta_3 \xi_L^2 - \frac{3}{1024}\,\zeta_4 \xi_L^2 \\
& - \frac{5}{256}\,\zeta_5 \xi_L^2 - \frac{191}{3072}\xi_L^3 + \frac{1}{192}\,\zeta_3 \xi_L^3 + \frac{294793}{27648} l_{q\mu} - \frac{1301}{512}\,\zeta_3 l_{q\mu} \\
& + \frac{6509}{6144}\,\xi_L l_{q\mu} - \frac{59}{256}\,\zeta_3\,\xi_L l_{q\mu} + \frac{467}{2048}\xi_L^2 l_{q\mu} - \frac{9}{512}\,\zeta_3 \xi_L^2 l_{q\mu} \\
& + \frac{43}{1024}\xi_L^3 l_{q\mu} - \frac{5507}{2304} l_{q\mu}^2 - \frac{715}{3072}\,\xi_L l_{q\mu}^2 - \frac{61}{1024}\xi_L^2 l_{q\mu}^2 - \frac{3}{256}\xi_L^3 l_{q\mu}^2 \\
& + \frac{121}{576} l_{q\mu}^3 + \frac{31}{1536}\,\xi_L l_{q\mu}^3 + \frac{3}{512}\xi_L^2 l_{q\mu}^3 + \frac{1}{768}\xi_L^3 l_{q\mu}^3 \Bigg] \\
+ \quad & C_F\, C_A\, T\, n_f \Bigg[\frac{16381}{1944} - \frac{193}{144}\,\zeta_3 + \frac{3}{8}\,\zeta_4 + \frac{2773}{4608}\,\xi_L - \frac{3}{16}\,\zeta_3\,\xi_L - \frac{5081}{864} l_{q\mu} \\
& + \frac{1}{8}\,\zeta_3 l_{q\mu} - \frac{251}{768}\,\xi_L l_{q\mu} + \frac{1}{16}\,\zeta_3\,\xi_L l_{q\mu} + \frac{887}{576} l_{q\mu}^2 + \frac{25}{384}\,\xi_L l_{q\mu}^2 \\
& - \frac{11}{72} l_{q\mu}^3 - \frac{1}{192}\,\xi_L l_{q\mu}^3 \Bigg] \\
+ \quad & C_F T^2\, n_f^2 \Bigg[-\frac{191}{243} - \frac{1}{18}\,\zeta_3 + \frac{73}{108} l_{q\mu} - \frac{2}{9} l_{q\mu}^2 + \frac{1}{36} l_{q\mu}^3 \Bigg] , \qquad \text{(A.3)}
\end{aligned}
$$

$$
\Sigma_v^{(1)} = \quad C_F \left[\frac{1}{4}\,\xi_L - \frac{1}{4}\,\xi_L l_{q\mu} \right] , \qquad \text{(A.4)}
$$

$$
\begin{aligned}
\Sigma_v^{(2)} = \quad & C_F^2 \left[-\frac{5}{128} + \frac{3}{32} l_{q\mu} - \frac{1}{16}\xi_L^2 l_{q\mu} + \frac{1}{32}\xi_L^2 l_{q\mu}^2 \right] \\
+ \quad & C_F\, C_A \Bigg[\frac{41}{64} - \frac{3}{16}\,\zeta_3 + \frac{13}{32}\,\xi_L - \frac{3}{16}\,\zeta_3\,\xi_L + \frac{9}{128}\xi_L^2 - \frac{25}{64} l_{q\mu} \\
& - \frac{7}{32}\,\xi_L l_{q\mu} - \frac{3}{64}\xi_L^2 l_{q\mu} + \frac{3}{64}\,\xi_L l_{q\mu}^2 + \frac{1}{64}\xi_L^2 l_{q\mu}^2 \Bigg] \\
+ \quad & C_F\, T\, n_f \left[-\frac{7}{32} + \frac{1}{8} l_{q\mu} \right] , \qquad \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_v^{(3)} = \quad & C_F^3 \Bigg[-\frac{73}{768} + \frac{7}{512}\,\xi_L - \frac{1}{96}\,\zeta_3 \xi_L^3 - \frac{3}{128} l_{q\mu} + \frac{17}{512}\,\xi_L l_{q\mu} - \frac{3}{128}\,\xi_L l_{q\mu}^2 \\
& + \frac{1}{128}\xi_L^3 l_{q\mu}^2 - \frac{1}{384}\xi_L^3 l_{q\mu}^3 \Bigg] \\
+ \quad & C_F^2\, C_A \Bigg[-\frac{997}{1536} + \frac{11}{16}\,\zeta_3 + \frac{3}{32}\,\zeta_4 - \frac{5}{16}\,\zeta_5 + \frac{1}{16}\,\xi_L - \frac{17}{64}\,\zeta_3\,\xi_L + \frac{5}{16}\,\zeta_5\,\xi_L \\
& + \frac{3}{128}\xi_L^2 - \frac{1}{512}\xi_L^3 + \frac{1}{64}\,\zeta_3 \xi_L^3 + \frac{121}{192} l_{q\mu} - \frac{3}{16}\,\zeta_3 l_{q\mu} - \frac{33}{128}\,\xi_L l_{q\mu} \\
& + \frac{3}{64}\,\zeta_3\,\xi_L l_{q\mu} - \frac{17}{128}\xi_L^2 l_{q\mu} + \frac{3}{64}\,\zeta_3 \xi_L^2 l_{q\mu} - \frac{11}{512}\xi_L^3 l_{q\mu} - \frac{11}{128} l_{q\mu}^2 \\
& + \frac{25}{256}\,\xi_L l_{q\mu}^2 + \frac{17}{256}\xi_L^2 l_{q\mu}^2 + \frac{1}{64}\xi_L^3 l_{q\mu}^2 - \frac{3}{256}\xi_L^2 l_{q\mu}^3 - \frac{1}{256}\xi_L^3 l_{q\mu}^3 \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad C_F^2\,T\,n_f\left[-\frac{79}{384}+\frac{1}{4}\,\zeta_3-\frac{3}{128}\,\xi_L-\frac{7}{96}l_{q\mu}+\frac{11}{128}\,\xi_L l_{q\mu}+\frac{1}{32}l_{q\mu}^2-\frac{1}{32}\,\xi_L l_{q\mu}^2\right]\\
&+ \quad C_F C_A^2\left[\frac{159257}{41472}-\frac{3139}{1536}\,\zeta_3-\frac{69}{1024}\,\zeta_4+\frac{165}{256}\,\zeta_5+\frac{39799}{36864}\,\xi_L-\frac{35}{64}\,\zeta_3\,\xi_L\right.\\
&\qquad +\frac{3}{512}\,\zeta_4\,\xi_L+\frac{5}{128}\,\zeta_5\,\xi_L+\frac{787}{4096}\xi_L^2-\frac{39}{512}\,\zeta_3\xi_L^2+\frac{3}{1024}\,\zeta_4\xi_L^2\\
&\qquad +\frac{5}{256}\,\zeta_5\xi_L^2+\frac{55}{1536}\xi_L^3-\frac{1}{192}\,\zeta_3\xi_L^3-\frac{19979}{9216}l_{q\mu}+\frac{245}{512}\,\zeta_3 l_{q\mu}\\
&\qquad -\frac{4393}{6144}\,\xi_L l_{q\mu}+\frac{59}{256}\,\zeta_3\,\xi_L l_{q\mu}-\frac{295}{2048}\xi_L^2 l_{q\mu}+\frac{9}{512}\,\zeta_3\xi_L^2 l_{q\mu}\\
&\qquad -\frac{27}{1024}\xi_L^3 l_{q\mu}+\frac{275}{768}l_{q\mu}^2+\frac{529}{3072}\,\xi_L l_{q\mu}^2+\frac{43}{1024}\xi_L^2 l_{q\mu}^2+\frac{1}{128}\xi_L^3 l_{q\mu}^2\\
&\qquad \left.-\frac{31}{1536}\,\xi_L l_{q\mu}^3-\frac{3}{512}\xi_L^2 l_{q\mu}^3-\frac{1}{768}\xi_L^3 l_{q\mu}^3\right]\\
&+ \quad C_F\,C_A\,T\,n_f\left[-\frac{11887}{5184}+\frac{13}{48}\,\zeta_3-\frac{1723}{4608}\,\xi_L+\frac{1}{8}\,\zeta_3\,\xi_L+\frac{191}{144}l_{q\mu}-\frac{1}{8}\,\zeta_3 l_{q\mu}\right.\\
&\qquad \left.+\frac{175}{768}\,\xi_L l_{q\mu}-\frac{1}{16}\,\zeta_3\,\xi_L l_{q\mu}-\frac{47}{192}l_{q\mu}^2-\frac{19}{384}\,\xi_L l_{q\mu}^2+\frac{1}{192}\,\xi_L l_{q\mu}^3\right]\\
&+ \quad C_F T^2\,n_f^2\left[\frac{785}{2592}-\frac{13}{72}l_{q\mu}+\frac{1}{24}l_{q\mu}^2\right]. \qquad \text{(A.6)}
\end{aligned}
$$

B Conversion Functions

The full gauge dependent conversion factors read[7]

$$
C_2^{\mathrm{RI}\;(1)} = \quad C_F\left[-\frac{1}{8}\,\xi_L\right], \qquad \text{(B.1)}
$$

$$
\begin{aligned}
C_2^{\mathrm{RI}\;(2)} = &\quad C_A\,C_F\left[-\frac{57}{128}+\frac{3}{16}\,\zeta_3-\frac{19}{64}\,\xi_L+\frac{3}{16}\,\zeta_3\,\xi_L-\frac{3}{64}\xi_L^2\right]\\
&+ \quad C_F^2\left[-\frac{1}{128}+\frac{3}{64}\xi_L^2\right] \quad + \quad C_F\,n_f\,T\left[\frac{5}{32}\right], \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
C_2^{\mathrm{RI}\;(3)} = &\quad C_A^2\,C_F\left[-\frac{457217}{165888}+\frac{5543}{3072}\,\zeta_3+\frac{69}{1024}\,\zeta_4-\frac{165}{256}\,\zeta_5-\frac{6655}{9216}\,\xi_L\right.\\
&\qquad +\frac{221}{512}\,\zeta_3\,\xi_L-\frac{3}{512}\,\zeta_4\,\xi_L-\frac{5}{128}\,\zeta_5\,\xi_L-\frac{123}{1024}\xi_L^2+\frac{69}{1024}\,\zeta_3\xi_L^2\\
&\qquad \left.-\frac{3}{1024}\,\zeta_4\xi_L^2-\frac{5}{256}\,\zeta_5\xi_L^2-\frac{139}{6144}\xi_L^3+\frac{1}{192}\,\zeta_3\xi_L^3\right]\\
&+ \quad C_A C_F^2\left[\frac{171}{512}-\frac{19}{32}\,\zeta_3-\frac{3}{32}\,\zeta_4+\frac{5}{16}\,\zeta_5+\frac{91}{512}\,\xi_L+\frac{25}{128}\,\zeta_3\,\xi_L\right.\\
&\qquad \left.-\frac{5}{16}\,\zeta_5\,\xi_L+\frac{15}{128}\xi_L^2-\frac{9}{128}\,\zeta_3\xi_L^2+\frac{25}{1024}\xi_L^3-\frac{1}{64}\,\zeta_3\xi_L^3\right]\\
&+ \quad C_F^3\left[\frac{41}{384}-\frac{29}{1024}\,\xi_L-\frac{5}{512}\xi_L^3+\frac{1}{96}\,\zeta_3\xi_L^3\right]
\end{aligned}
$$

[7]Note that these results are expanded in $a_s=\frac{\alpha_s}{\pi}$ and not in $\frac{\alpha_s}{4\pi}$ as in Eqs. (34) to (37).

$$
\begin{aligned}
&+ \quad C_A\, C_F\, n_f\, T \left[\frac{8449}{5184} - \frac{5}{24}\,\zeta_3 + \frac{599}{2304}\,\xi_L - \frac{3}{32}\,\zeta_3\,\xi_L\right] \\
&+ \quad C_F^2\, n_f\, T \left[\frac{31}{128} - \frac{1}{4}\,\zeta_3 - \frac{15}{256}\,\xi_L\right] \quad + \quad C_F\, n_f^2 T^2 \left[-\frac{551}{2592}\right], \qquad \text{(B.3)}
\end{aligned}
$$

$$
C_m^{\mathrm{RI}\ (1)} = \qquad C_F \left[-1 - \frac{3}{8}\,\xi_L\right], \qquad \text{(B.4)}
$$

$$
\begin{aligned}
C_m^{\mathrm{RI}\ (2)} = \qquad & C_A\, C_F \left[-\frac{85}{24} + \frac{9}{8}\,\zeta_3 - \frac{21}{64}\,\xi_L - \frac{9}{128}\xi_L^2\right] \\
+ \quad & C_F^2 \left[\frac{25}{128} - \frac{3}{4}\,\zeta_3 + \frac{3}{8}\,\xi_L + \frac{3}{32}\xi_L^2\right] \quad + \quad C_F\, n_f\, T \left[\frac{89}{96}\right], \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
C_m^{\mathrm{RI}\ (3)} = \qquad & C_A^2\, C_F \left[-\frac{7259479}{497664} + \frac{64591}{9216}\,\zeta_3 - \frac{15}{16}\,\zeta_5 - \frac{4409}{4096}\,\xi_L + \frac{165}{512}\,\zeta_3\,\xi_L \right.\\
& \qquad \left. - \frac{885}{4096}\xi_L^2 + \frac{27}{1024}\,\zeta_3\xi_L^2 - \frac{81}{2048}\xi_L^3\right] \\
+ \quad & C_A C_F^2 \left[\frac{21133}{4608} - \frac{245}{48}\,\zeta_3 + \frac{5}{16}\,\zeta_5 + \frac{1771}{1024}\,\xi_L - \frac{13}{16}\,\zeta_3\,\xi_L \right.\\
& \qquad \left. + \frac{69}{256}\xi_L^2 - \frac{3}{64}\,\zeta_3\xi_L^2 + \frac{3}{64}\xi_L^3\right] \\
+ \quad & C_F^3 \left[-\frac{409}{96} + \frac{29}{32}\,\zeta_3 + \frac{15}{8}\,\zeta_5 - \frac{93}{1024}\,\xi_L - \frac{3}{32}\xi_L^2 + \frac{3}{32}\,\zeta_3\xi_L^2 \right.\\
& \qquad \left. - \frac{3}{128}\xi_L^3\right] \\
+ \quad & C_A\, C_F\, n_f\, T \left[\frac{105701}{15552} - \frac{163}{144}\,\zeta_3 + \frac{3}{8}\,\zeta_4 + \frac{175}{512}\,\xi_L - \frac{3}{32}\,\zeta_3\,\xi_L\right] \\
+ \quad & C_F^2\, n_f\, T \left[\frac{263}{144} - \frac{2}{3}\,\zeta_3 - \frac{3}{8}\,\zeta_4 - \frac{95}{256}\,\xi_L + \frac{1}{8}\,\zeta_3\,\xi_L\right] \\
+ \quad & C_F\, n_f^2 T^2 \left[-\frac{4459}{7776} - \frac{1}{18}\,\zeta_3\right], \qquad \text{(B.6)}
\end{aligned}
$$

$$
C_2^{\mathrm{RI'}(1)} = \qquad C_F \left[-\frac{1}{4}\,\xi_L\right], \qquad \text{(B.7)}
$$

$$
\begin{aligned}
C_2^{\mathrm{RI'}(2)} = \qquad & C_A\, C_F \left[-\frac{41}{64} + \frac{3}{16}\,\zeta_3 - \frac{13}{32}\,\xi_L + \frac{3}{16}\,\zeta_3\,\xi_L - \frac{9}{128}\xi_L^2\right] \\
+ \quad & C_F^2 \left[\frac{5}{128} + \frac{1}{16}\xi_L^2\right] \quad + \quad C_F\, n_f\, T \left[\frac{7}{32}\right], \qquad \text{(B.8)}
\end{aligned}
$$

$$
\begin{aligned}
C_2^{\mathrm{RI'}(3)} = \qquad & C_A C_F^2 \left[\frac{997}{1536} - \frac{11}{16}\,\zeta_3 - \frac{3}{32}\,\zeta_4 + \frac{5}{16}\,\zeta_5 + \frac{33}{128}\,\xi_L + \frac{11}{64}\,\zeta_3\,\xi_L \right.\\
& \qquad \left. - \frac{5}{16}\,\zeta_5\,\xi_L + \frac{23}{128}\xi_L^2 - \frac{3}{32}\,\zeta_3\xi_L^2 + \frac{19}{512}\xi_L^3 - \frac{1}{64}\,\zeta_3\xi_L^3\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad C_A^2\, C_F \left[-\frac{159257}{41472} + \frac{3139}{1536}\,\zeta_3 + \frac{69}{1024}\,\zeta_4 - \frac{165}{256}\,\zeta_5 \right.\\
&\qquad - \frac{39799}{36864}\,\xi_L + \frac{35}{64}\,\zeta_3\,\xi_L - \frac{3}{512}\,\zeta_4\,\xi_L - \frac{5}{128}\,\zeta_5\,\xi_L\\
&\qquad - \frac{787}{4096}\xi_L^2 + \frac{39}{512}\,\zeta_3\xi_L^2 - \frac{3}{1024}\,\zeta_4\xi_L^2 - \frac{5}{256}\,\zeta_5\xi_L^2\\
&\qquad \left. - \frac{55}{1536}\xi_L^3 + \frac{1}{192}\,\zeta_3\xi_L^3 \right]\\
&+ \quad C_F^3 \left[\frac{73}{768} - \frac{17}{512}\,\xi_L - \frac{1}{64}\xi_L^3 + \frac{1}{96}\,\zeta_3\xi_L^3 \right]\\
&+ \quad C_A\, C_F\, n_f\, T \left[\frac{11887}{5184} - \frac{13}{48}\,\zeta_3 + \frac{1723}{4608}\,\xi_L - \frac{1}{8}\,\zeta_3\,\xi_L \right]\\
&+ \quad C_F^2\, n_f\, T \left[\frac{79}{384} - \frac{1}{4}\,\zeta_3 - \frac{11}{128}\,\xi_L \right] \quad + \quad C_F\, n_f^2 T^2 \left[-\frac{785}{2592} \right],
\end{aligned} \tag{B.9}
$$

$$
C_m^{\mathrm{RI'}(1)} = \quad C_F \left[-1 - \frac{1}{4}\,\xi_L \right], \tag{B.10}
$$

$$
\begin{aligned}
C_m^{\mathrm{RI'}(2)} = &\quad C_A\, C_F \left[-\frac{1285}{384} + \frac{9}{8}\,\zeta_3 - \frac{7}{32}\,\xi_L - \frac{3}{64}\xi_L^2 \right]\\
&+ \quad C_F^2 \left[\frac{19}{128} - \frac{3}{4}\,\zeta_3 + \frac{1}{4}\,\xi_L + \frac{1}{16}\xi_L^2 \right] \quad + \quad C_F\, n_f\, T \left[\frac{83}{96} \right],
\end{aligned} \tag{B.11}
$$

$$
\begin{aligned}
C_m^{\mathrm{RI'}(3)} = &\quad C_A^2\, C_F \left[-\frac{3360023}{248832} + \frac{31193}{4608}\,\zeta_3 - \frac{15}{16}\,\zeta_5 - \frac{4417}{6144}\,\xi_L + \frac{53}{256}\,\zeta_3\,\xi_L \right.\\
&\qquad \left. - \frac{295}{2048}\xi_L^2 + \frac{9}{512}\,\zeta_3\xi_L^2 - \frac{27}{1024}\xi_L^3 \right]\\
&+ \quad C_A C_F^2 \left[\frac{18781}{4608} - \frac{481}{96}\,\zeta_3 + \frac{5}{16}\,\zeta_5 + \frac{1771}{1536}\,\xi_L - \frac{43}{64}\,\zeta_3\,\xi_L \right.\\
&\qquad \left. + \frac{23}{128}\xi_L^2 - \frac{3}{64}\,\zeta_3\xi_L^2 + \frac{1}{32}\xi_L^3 \right]\\
&+ \quad C_F^3 \left[-\frac{3227}{768} + \frac{29}{32}\,\zeta_3 + \frac{15}{8}\,\zeta_5 - \frac{31}{512}\,\xi_L - \frac{3}{32}\,\zeta_3\,\xi_L - \frac{1}{16}\xi_L^2 \right.\\
&\qquad \left. + \frac{3}{32}\,\zeta_3\xi_L^2 - \frac{1}{64}\xi_L^3 \right]\\
&+ \quad C_A\, C_F\, n_f\, T \left[\frac{95387}{15552} - \frac{77}{72}\,\zeta_3 + \frac{3}{8}\,\zeta_4 + \frac{175}{768}\,\xi_L - \frac{1}{16}\,\zeta_3\,\xi_L \right]\\
&+ \quad C_F^2\, n_f\, T \left[\frac{1109}{576} - \frac{2}{3}\,\zeta_3 - \frac{3}{8}\,\zeta_4 - \frac{95}{384}\,\xi_L + \frac{1}{8}\,\zeta_3\,\xi_L \right]\\
&+ \quad C_F\, n_f^2 T^2 \left[-\frac{3757}{7776} - \frac{1}{18}\,\zeta_3 \right].
\end{aligned} \tag{B.12}
$$

C Anomalous Dimensions

The quark mass and field anomalous dimensions for general $SU(N)$ and for the $\overline{\mathrm{MS}}$, RI and RI' schemes are listed below. See Eq. (46) for the conventions. For the $\overline{\mathrm{MS}}$ case also the field anomalous dimension γ_2 is given for general gauge and $SU(N)$, while for the RI and RI' only the Landau-gauge formulae are given (for which all $\gamma_2^{(0)}$ are 0):

$$\gamma_m^{(0)} = \frac{N^2-1}{N}\left\{ \quad \frac{3}{8} \quad \right\}, \tag{C.1}$$

$$\gamma_m^{(1)} = \frac{N^2-1}{16N^2}\left\{ \quad \left[-\frac{3}{8}+\frac{203}{24}N^2\right] \quad + \quad n_f\left[-\frac{5}{6}N\right] \quad \right\}, \tag{C.2}$$

$$\begin{aligned}\gamma_m^{(2)} = \frac{N^2-1}{64N^3}\Bigg\{ & \quad \left[\frac{129}{16}-\frac{129}{16}N^2+\frac{11413}{216}N^4\right] \\ & + \quad n_f\left[\frac{23}{4}N-\frac{1177}{108}N^3-6N\,\zeta_3-6N^3\,\zeta_3\right] \\ & + \quad n_f^2\left[-\frac{35}{54}N^2\right] \quad \Bigg\},\end{aligned} \tag{C.3}$$

$$\begin{aligned}\gamma_m^{(3)} = \frac{N^2-1}{256N^4}\Bigg\{ & \quad \Bigg[\frac{1261}{128}+\frac{50047}{384}N^2-\frac{66577}{1152}N^4+\frac{460151}{1152}N^6+21\,\zeta_3 \\ & \quad -\frac{47}{2}N^2\,\zeta_3+52N^4\,\zeta_3+\frac{1157}{18}N^6\,\zeta_3-110N^4\,\zeta_5-110N^6\,\zeta_5\Bigg] \\ & + \quad n_f\Bigg[\frac{37}{6}N+\frac{10475}{216}N^3-\frac{11908}{81}N^5-\frac{111}{2}N\,\zeta_3-85N^3\,\zeta_3-\frac{889}{6}N^5\,\zeta_3 \\ & \quad +33N^3\,\zeta_4+33N^5\,\zeta_4-30N\,\zeta_5+50N^3\,\zeta_5+80N^5\,\zeta_5\Bigg] \\ & + \quad n_f^2\left[-\frac{19}{27}N^2+\frac{899}{324}N^4+10N^2\,\zeta_3+10N^4\,\zeta_3-6N^2\,\zeta_4-6N^4\,\zeta_4\right] \\ & + \quad n_f^3\left[-\frac{83}{162}N^3+\frac{8}{9}N^3\,\zeta_3\right] \quad \Bigg\},\end{aligned} \tag{C.4}$$

$$\gamma_m^{\mathrm{RI}\;(0)} = \frac{N^2-1}{2N}\left\{ \quad \frac{3}{4} \quad \right\}, \tag{C.5}$$

$$\gamma_m^{\mathrm{RI}\;(1)} = \frac{N^2-1}{16N^2}\left\{ \quad \left[-\frac{3}{8}+\frac{379}{24}N^2\right] \quad + \quad n_f\left[-\frac{13}{6}N\right] \quad \right\}, \tag{C.6}$$

$$\begin{aligned}\gamma_m^{\mathrm{RI}\;(2)} = \frac{N^2-1}{64N^3}\Bigg\{ & \quad \left[\frac{129}{16}-17N^2-22\,\zeta_3N^2+\frac{126239}{432}N^4-44\,\zeta_3N^4\right] \\ & + \quad n_f\left[\frac{75}{8}N-2\,\zeta_3N-\frac{18611}{216}N^3+2\,\zeta_3N^3\right] \\ & + \quad n_f^2\left[\frac{116}{27}N^2\right] \quad \Bigg\},\end{aligned} \tag{C.7}$$

$$\gamma_m^{\mathrm{RI}\ (3)} = \frac{N^2-1}{256N^4}\Bigg\{ \left[\frac{1261}{128} + \frac{198679}{384}N^2 - \frac{778421}{1152}N^4 + \frac{202580155}{31104}N^6 + 21\,\zeta_3 \right. \\ \left. - \frac{149}{4}N^2\,\zeta_3 - \frac{4133}{6}N^4\,\zeta_3 - \frac{59269}{32}N^6\,\zeta_3 - 165N^2\,\zeta_5 + 275N^4\,\zeta_5\right] \\ + \quad n_f \left[-\frac{3175}{48}N + \frac{206575}{432}N^3 - \frac{7671073}{2592}N^5 - 59N\,\zeta_3 - \frac{22}{3}N^3\,\zeta_3 \right. \\ \left. + \frac{7767}{16}N^5\,\zeta_3 - 20N^3\,\zeta_5 + 60N^5\,\zeta_5\right] \\ + \quad n_f^2 \left[-\frac{5767}{108}N^2 + \frac{113747}{324}N^4 + \frac{62}{3}N^2\,\zeta_3 - 32N^4\,\zeta_3\right] \\ + \quad n_f^3 \left[-\frac{2354}{243}N^3\right] \Bigg\}, \tag{C.8}$$

$$\gamma_m^{\mathrm{RI'}(0)} = \frac{N^2-1}{2N}\Bigg\{ \frac{3}{4} \Bigg\}, \tag{C.9}$$

$$\gamma_m^{\mathrm{RI'}(1)} = \frac{N^2-1}{16N^2}\Bigg\{ \left[-\frac{3}{8} + \frac{379}{24}N^2\right] \quad + \quad n_f \left[-\frac{13}{6}N\right] \Bigg\}, \tag{C.10}$$

$$\gamma_m^{\mathrm{RI'}(2)} = \frac{N^2-1}{64N^3}\Bigg\{ \left[\frac{129}{16} - \frac{147}{8}N^2 - 22\,\zeta_3 N^2 + \frac{121883}{432}N^4 - 44\,\zeta_3 N^4\right] \\ + \quad n_f \left[\frac{77}{8}N - 2\,\zeta_3 N - \frac{17819}{216}N^3 + 2\,\zeta_3 N^3\right] \\ + \quad n_f^2 \left[\frac{107}{27}N^2\right] \Bigg\}, \tag{C.11}$$

$$\gamma_m^{\mathrm{RI'}(3)} = \frac{N^2-1}{256N^4}\Bigg\{ \left[\frac{1261}{128} + 21\,\zeta_3 + \frac{198283}{384}N^2 - \frac{149}{4}\,\zeta_3 N^2 - 165\,\zeta_5 N^2 \right. \\ - \frac{844829}{1152}N^4 - \frac{2017}{3}\,\zeta_3 N^4 + 275\,\zeta_5 N^4 \\ \left. + \frac{191436121}{31104}N^6 - \frac{28551}{16}\,\zeta_3 N^6\right] \\ + \quad n_f \left[-\frac{199}{3}N - 59\,\zeta_3 N + \frac{211669}{432}N^3 - \frac{31}{3}\,\zeta_3 N^3 - 20\,\zeta_5 N^3 \right. \\ \left. - \frac{1794277}{648}N^5 + \frac{3697}{8}\,\zeta_3 N^5 + 60\,\zeta_5 N^5\right] \\ + \quad n_f^2 \left[-\frac{1444}{27}N^2 + \frac{62}{3}\,\zeta_3 N^2 + \frac{25946}{81}N^4 - 30\,\zeta_3 N^4\right] \\ + \quad n_f^3 \left[-\frac{2003}{243}N^3\right] \Bigg\}, \tag{C.12}$$

$$\gamma_2^{(0)} = \frac{N^2-1}{N}\Bigg\{ \quad \left[\frac{1}{8}\,\xi_L\right] \quad \Bigg\}, \tag{C.13}$$

$$\gamma_2^{(1)} = \frac{N^2-1}{16N^2}\Bigg\{ \quad \left[\frac{3}{8}+\frac{11}{4}N^2+N^2\,\xi_L+\frac{1}{8}N^2\xi_L^2\right] \quad + \quad n_f\left[-\frac{1}{2}N\right] \quad \Bigg\}, \tag{C.14}$$

$$\begin{aligned}\gamma_2^{(2)} = \frac{N^2-1}{64N^3}\Bigg\{ \quad & \left[\frac{3}{16}+\frac{137}{16}N^2+\frac{6635}{288}N^4+\frac{263}{64}N^4\,\xi_L+\frac{39}{64}N^4\xi_L^2\right. \\ & \left. +\frac{5}{32}N^4\xi_L^3-3N^2\,\zeta_3-\frac{21}{16}N^4\,\zeta_3+\frac{3}{8}N^4\,\xi_L\,\zeta_3+\frac{3}{16}N^4\xi_L^2\,\zeta_3\right] \\ + \quad & n_f\left[-\frac{3}{8}N-\frac{547}{72}N^3-\frac{17}{16}N^3\,\xi_L\right]+n_f^2\left[\frac{5}{18}N^2\right] \quad \Bigg\},\end{aligned} \tag{C.15}$$

$$\begin{aligned}\gamma_2^{(3)} = \frac{N^2-1}{256N^4}\Bigg\{ \quad & \left[\frac{1027}{128}+\frac{11281}{384}N^2+\frac{86017}{1152}N^4+\frac{1785121}{10368}N^6-\frac{5}{8}N^4\,\xi_L+\frac{1644899}{62208}N^6\,\xi_L\right. \\ & +\frac{6467}{2304}N^6\xi_L^2+\frac{149}{192}N^6\xi_L^3+\frac{19}{128}N^6\xi_L^4+25\,\zeta_3+31N^2\,\zeta_3 \\ & -\frac{4287}{64}N^4\,\zeta_3-\frac{1103}{64}N^6\,\zeta_3+\frac{115}{16}N^4\,\xi_L\,\zeta_3+\frac{2761}{192}N^6\,\xi_L\,\zeta_3+\frac{11}{32}N^4\xi_L^2\,\zeta_3 \\ & +\frac{289}{96}N^6\xi_L^2\,\zeta_3-\frac{3}{16}N^4\xi_L^3\,\zeta_3+\frac{19}{64}N^6\xi_L^3\,\zeta_3-\frac{7}{64}N^4\xi_L^4\,\zeta_3-\frac{1}{64}N^6\xi_L^4\,\zeta_3 \\ & +\frac{33}{2}N^4\,\zeta_4+\frac{231}{32}N^6\,\zeta_4-\frac{97}{64}N^6\,\xi_L\,\zeta_4-\frac{17}{32}N^6\xi_L^2\,\zeta_4-\frac{3}{64}N^6\xi_L^3\,\zeta_4 \\ & -40\,\zeta_5-60N^2\,\zeta_5+\frac{1945}{64}N^4\,\zeta_5-\frac{1375}{128}N^6\,\zeta_5-\frac{65}{8}N^4\,\xi_L\,\zeta_5 \\ & \left.-\frac{95}{8}N^6\,\xi_L\,\zeta_5-\frac{35}{32}N^4\xi_L^2\,\zeta_5-\frac{75}{64}N^6\xi_L^2\,\zeta_5+\frac{5}{64}N^4\xi_L^4\,\zeta_5+\frac{5}{128}N^6\xi_L^4\,\zeta_5\right] \\ + \quad & n_f\left[\frac{307}{12}N-\frac{1555}{144}N^3-\frac{35641}{432}N^5+\frac{767}{96}N^3\,\xi_L-\frac{161347}{15552}N^5\,\xi_L\right. \\ & -\frac{109}{288}N^5\xi_L^2-4N\,\zeta_3+8N^3\,\zeta_3-\frac{35}{8}N^5\,\zeta_3-\frac{11}{2}N^3\,\xi_L\,\zeta_3 \\ & -\frac{15}{4}N^5\,\xi_L\,\zeta_3-\frac{7}{24}N^5\xi_L^2\,\zeta_3-3N^3\,\zeta_4-\frac{21}{16}N^5\,\zeta_4-\frac{3}{2}N^3\,\xi_L\,\zeta_4 \\ & \left.-\frac{1}{2}N^5\,\xi_L\,\zeta_4+\frac{1}{16}N^5\xi_L^2\,\zeta_4-20N^3\,\zeta_5\right] \\ + \quad & n_f^2\left[-\frac{19}{9}N^2+\frac{445}{72}N^4-\frac{269}{486}N^4\,\xi_L+2N^2\,\zeta_3+2N^4\,\zeta_3+\frac{2}{3}N^4\,\xi_L\,\zeta_3\right] \\ + \quad & n_f^3\left[\frac{35}{162}N^3\right] \quad \Bigg\},\end{aligned} \tag{C.16}$$

$$\gamma_2^{\mathrm{RI}\,(1)} = \frac{N^2-1}{16N^2}\Bigg\{ \quad \left[\frac{3}{8}+\frac{11}{4}N^2\right] \quad + \quad n_f\left[-\frac{1}{2}N\right] \quad \Bigg\}, \tag{C.17}$$

$$\gamma_2^{\mathrm{RI}\ (2)} = \frac{N^2-1}{64N^3}\Bigg\{ \left[\frac{3}{16} + \frac{25}{3}N^2 + \frac{14225}{288}N^4 - 3N^2\,\zeta_3 - \frac{197}{16}N^4\,\zeta_3\right] + n_f\left[-\frac{1}{3}N - \frac{611}{36}N^3 + 2N^3\,\zeta_3\right] + n_f^2\left[\frac{10}{9}N^2\right] \Bigg\}, \quad (C.18)$$

$$\begin{aligned}\gamma_2^{\mathrm{RI}\ (3)} = \frac{N^2-1}{256N^4}\Bigg\{ & \left[\frac{1027}{128} + \frac{7673}{384}N^2 + \frac{174565}{1152}N^4 + \frac{3993865}{3456}N^6 + 25\,\zeta_3 + 31N^2\,\zeta_3\right. \\ & - \frac{10975}{64}N^4\,\zeta_3 - \frac{111719}{192}N^6\,\zeta_3 - 40\,\zeta_5 - 60N^2\,\zeta_5 + \frac{5465}{64}N^4\,\zeta_5 \\ & \left. + \frac{20625}{128}N^6\,\zeta_5\right] \\ + & n_f\left[\frac{1307}{48}N + \frac{557}{144}N^3 - \frac{172793}{288}N^5 - 4N\,\zeta_3 + 2N^3\,\zeta_3\right. \\ & \left. + \frac{7861}{48}N^5\,\zeta_3 - 30N^3\,\zeta_5 - \frac{125}{4}N^5\,\zeta_5\right] \\ + & n_f^2\left[-\frac{521}{72}N^2 + \frac{259}{3}N^4 + 6N^2\,\zeta_3 - \frac{26}{3}N^4\,\zeta_3\right] \\ + & n_f^3\left[-\frac{86}{27}N^3\right] \Bigg\}, \end{aligned} \quad (C.19)$$

$$\gamma_2^{\mathrm{RI'}(1)} = \frac{N^2-1}{16N^2}\Bigg\{ \left[\frac{3}{8} + \frac{11}{4}N^2\right] + n_f\left[-\frac{1}{2}N\right] \Bigg\}, \quad (C.20)$$

$$\gamma_2^{\mathrm{RI'}(2)} = \frac{N^2-1}{64N^3}\Bigg\{ \left[\frac{3}{16} + \frac{233}{24}N^2 + \frac{17129}{288}N^4 - 3N^2\,\zeta_3 - \frac{197}{16}N^4\,\zeta_3\right] + n_f\left[-\frac{7}{12}N - \frac{743}{36}N^3 + 2N^3\,\zeta_3\right] + n_f^2\left[\frac{13}{9}N^2\right] \Bigg\}, \quad (C.21)$$

$$\begin{aligned}\gamma_2^{\mathrm{RI'}(3)} = \frac{N^2-1}{256N^4}\Bigg\{ & \left[\frac{1027}{128} + \frac{8069}{384}N^2 + \frac{240973}{1152}N^4 + \frac{5232091}{3456}N^6 + 25\,\zeta_3 + 31N^2\,\zeta_3\right. \\ & - \frac{12031}{64}N^4\,\zeta_3 - \frac{124721}{192}N^6\,\zeta_3 - 40\,\zeta_5 - 60N^2\,\zeta_5 + \frac{5465}{64}N^4\,\zeta_5 \\ & \left. + \frac{20625}{128}N^6\,\zeta_5\right] \\ + & n_f\left[\frac{329}{12}N - \frac{1141}{144}N^3 - \frac{113839}{144}N^5 - 4N\,\zeta_3 + 5N^3\,\zeta_3\right. \\ & \left. + \frac{2245}{12}N^5\,\zeta_3 - 30N^3\,\zeta_5 - \frac{125}{4}N^5\,\zeta_5\right] \\ + & n_f^2\left[-\frac{515}{72}N^2 + \frac{1405}{12}N^4 + 6N^2\,\zeta_3 - \frac{32}{3}N^4\,\zeta_3\right] \\ + & n_f^3\left[-\frac{125}{27}N^3\right] \Bigg\}. \end{aligned} \quad (C.22)$$

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