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Quark mass effects to $\sigma(e^+e^- \to \text{hadrons})$ at $\mathcal{O}(\alpha_s^3)$†

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1. Introduction

The hadronic R ratio is one of the most important quantities of QCD. It is defined as the total cross section for hadronproduction in e^+e^- annihilation, compared to the production rate for muon pairs:

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \,. \tag{1}$$

Its inclusive character is a welcome feature both from the experimental and the theoretical point of view. For example, the calculation of $R(s)$ can be performed fully perturbatively, as long as bound state effects which emerge close to threshold can be neglected.

The standard way to compute the hadronic R ratio is to apply the optical theorem which relates it to the imaginary part of the polarization function:

$$R(s) = 12\pi \mathrm{Im}\Pi(s + i\epsilon) \,. \tag{2}$$

Examples for diagrams contributing to $\Pi(q^2)$ up to order α_s^3 are shown in Fig. 1. The analytic evaluation of such diagrams is only possible up to two-loop order, if a non-vanishing quark mass is assumed. At three-loop order, only the massless case can be computed fully analytically. Nevertheless, mass effects at $\mathcal{O}(\alpha_s^2)$ could be evaluated to very high accuracy, using two different approximation methods. One is based on asymptotic expansions [1], yielding the result in terms of a series in m^2/q^2 times logarithmic terms of the form $\ln(-q^2/\mu^2)$ and $\ln(-q^2/m^2)$, where q^2 is the renormalization scale [2]. The second method uses the moments of $\Pi(q^2)$, defined as $C_n = (\partial^n/\partial(q^2)^n)\Pi(q^2)|_{q^2=0}$, to reconstruct $\Pi(q^2)$ in the full complex plane. The result for $R(s)$ is therefore valid over the full energy range in this case [3].

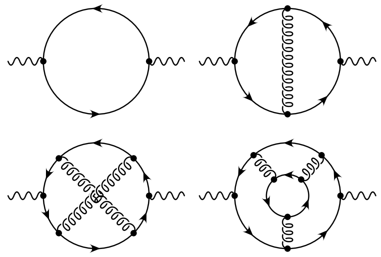

Figure 1. Sample diagrams contributing to $\Pi(q^2)$ up to four-loop level.

At $\mathcal{O}(\alpha_s^3)$, however, either of the two methods above inevitably leads to four-loop diagrams whose evaluation is possible only for a limited number of special cases at the moment. Nevertheless, besides the massless approximation [4] which could be reduced to a three-loop calculation using a method called infrared re-arrangement [5], also the quadratic mass corrections at $\mathcal{O}(\alpha_s^3)$ are available [6]. They were obtained by a pure renormalization group (RG) analysis using the terms at $\mathcal{O}(\alpha_s^2)$ as input information.

At lower order in α_s, the quartic (m^4) mass corrections turn out to be quite sizable and are thus needed to provide a good approximation to the full result over a reasonably large energy range. This was the main motivation for us to evaluate these corrections at $\mathcal{O}(\alpha_s^3)$. The method which was followed has been developed in [7]. At that time it was used to determine for the first time the quartic corrections at $\mathcal{O}(\alpha_s^2)$. The strategy is to evaluate the coefficient functions and the matrix elements of

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the operator product expansion (OPE) for $\Pi(q^2)$ to (in our case) the three-loop approximation and to use the RG equation in order to extend the validity of the result to $\mathcal{O}(\alpha_s^3)$.

2. Operator product expansion (OPE)

The polarization function is defined as the vacuum expectation value of the time ordered product of two currents, $j_\mu = \bar\psi\gamma_\mu\psi$, where ψ be a quark field with mass m. In the limit of large q^2, $\Pi(q^2)$ can be written asymptotically as an OPE:

$$\begin{aligned}\Pi_{\mu\nu}(q) &= i\int \mathrm{d}^4x e^{iq\cdot x}\langle 0|\mathrm{T}j_\mu(x)j_\nu(0)|0\rangle \\ &\stackrel{-q^2\to\infty}{\to} \sum_n C_{n,\mu\nu}\langle\mathcal{O}_n\rangle\,, \end{aligned}\tag{3}$$

where $\langle\mathcal{O}_n\rangle \equiv \langle 0|\mathcal{O}_n|0\rangle$. Only Lorentz scalar operators contribute due to the external vacuum states. The function $\Pi(q^2)$ in (2) is defined as the transversal part of $\Pi_{\mu\nu}(q)$,

$$\Pi(q^2) = \frac{1}{q^2}\frac{1}{D-1}\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\Pi_{\mu\nu}(q)\,. \tag{4}$$

An analogous definition is used to define transversal coefficient functions C_n. The operators up to dimension four are given by

$$\begin{aligned} &\mathcal{O}^{(0)} = \mathbf{1}\,, \qquad \mathcal{O}^{(2)} = \mathbf{m^2}\,, \\ &\mathcal{O}_1^{(4)} = G_{\mu\nu}^2\,, \quad \mathcal{O}_2^{(4)} = m\bar\psi\psi\,, \quad \mathcal{O}_3^{(4)} = \mathbf{m^4} \end{aligned}\tag{5}$$

(the superscript "(4)" will be dropped in what follows). In addition to the renormalization of m and α_s, one has to take into account operator mixing according to

$$\mathcal{O}_n = \sum_m Z_{nm}\mathcal{O}_m^{\mathrm{B}}\,. \tag{6}$$

In the $\overline{\mathrm{MS}}$ scheme which we will use throughout this paper, operators of different mass dimension do not mix under renormalization. For the dimension-4 operators the renormalization matrix Z_{nm} has been evaluated in [8]. It was expressed through the renormalization constants for m and α_s, plus the one for the QCD vacuum energy, Z_0. If no normal ordering is applied to the operators, the perturbative expressions for their vacuum expectation values are not necessarily zero. They lead to tadpole diagrams, examples of which are shown in Fig. 2. The matrix elements for the "trivial" operators are given by $\langle\mathbf{m}^{2n}\rangle = m^{2n}$ to all orders of perturbation theory. The ones for $\mathcal{O}_1$ and $\mathcal{O}_2$ can be obtained up to $\mathcal{O}(\alpha_s^2)$ by performing a

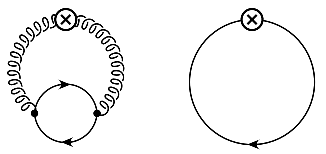

Figure 2. Sample diagrams contributing to $\langle\mathcal{O}_1\rangle$ and $\langle\mathcal{O}_2\rangle$.

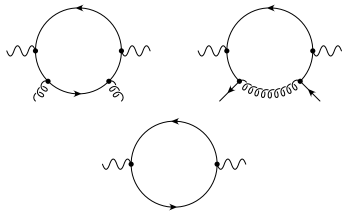

Figure 3. Sample diagrams contributing to C_1, C_2, and C_3.

three-loop calculation, as is evident from a look at the diagrams in Fig. 2.

Concerning the coefficient functions it has been shown in [9] that they depend only on the external momentum if minimal subtraction is used as the renormalization scheme. The most elaborate method to reduce their evaluation to massless propagator diagrams is called the "method of projectors" [10]. It leads to diagrams like the ones shown in Fig. 3, where it is understood that certain derivatives w.r.t. the quark mass and the momentum carried by the external quark and gluon lines are to be applied to the integrand. Before integration, however, all masses and external momenta (except q) can be set to zero. A glance at these diagrams shows that a three-loop calculation yields C_1 and C_2 with $\mathcal{O}(\alpha_s^3)$, but C_3 only with $\mathcal{O}(\alpha_s^2)$ accuracy. Furthermore, one realizes that the imaginary parts arising from C_1 and C_2 start to be non-zero only at $\mathcal{O}(\alpha_s)$. Thus the α_s^2 calculation for the matrix elements mentioned before is sufficient to determine the imaginary part of the products $C_1\langle\mathcal{O}_1\rangle$ and $C_2\langle\mathcal{O}_2\rangle$ to $\mathcal{O}(\alpha_s^3)$.

The crucial step to reconstruct the α_s^3 terms of C_3 is now to employ RG invariance of the dimension-4 piece of the OPE:

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\sum_{n=1}^{3} C_n\mathcal{O}_n = 0\,. \tag{7}$$

The effect of $\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}$ on the $\mathcal{O}_n$ is determined by (6):

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\mathcal{O}_n = \sum_m \gamma_{nm}\mathcal{O}_n\,, \tag{8}$$

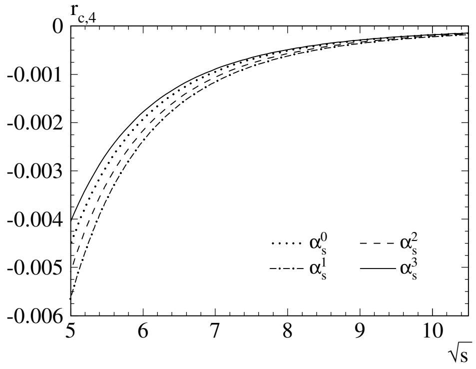

Figure 4. Quartic mass corrections to charm pair production.

where γ_{mn} is the anomalous dimension of the operators which is related to the renormalization matrix Z_{mn} in the standard way. For the coefficient functions, the derivative w.r.t. μ will be rewritten as

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} C_n = \left(\frac{\partial}{\partial L} + \alpha_s \beta \frac{\partial}{\partial \alpha_s}\right) C_n\,, \qquad (9)$$

with the QCD β function and $L = \ln(-\mu^2/q^2)$. Solving (7) for $\frac{\partial}{\partial L} C_3$ and keeping only terms proportional to $\mathcal{O}_3$, one obtains

$$\frac{\partial}{\partial L} C_3 = -\alpha_s \beta \frac{\partial}{\partial \alpha_s} C_3 - 4\gamma^m C_3 - \sum_{n=1}^{2} \gamma_{n3} C_n\,, \qquad (10)$$

where $\gamma_{33} = 4\gamma^m$ has been used (γ^m is the quark anomalous dimension). Since β and γ^m do not carry terms of $\mathcal{O}(\alpha_s^0)$, the r.h.s. of (10) is known to $\mathcal{O}(\alpha_s^3)$, and so is the l.h.s. The logarithmic terms (and thus the imaginary part) of C_3 may therefore be obtained by trivial integration.

3. Results

Let us consider the case of charm quark pair production. The general result for the terms of order $\alpha_s^3 m^4/s^2$ to $R(s)$ will be given elsewhere [11]. The upper limit of the energy region we are interested in is dictated by the $b\bar{b}$ threshold. The lower one is such that, on the one hand, the massless charm approximation fails, and on the other hand, one is sufficiently far above the $c\bar{c}$ threshold in order for the small mass expansion to be valid (see [2] for a detailed analysis on the convergence properties at $\mathcal{O}(\alpha_s^2)$). Fig. 4 shows the result for the contribution where the external photon couples directly to the charm quark. There is also a contribution arising from diagrams with only massless quarks coupling to the photon and the charm quarks emerging only through their coupling to gluons. In turns out, however, that the latter piece is tiny, so we will neglect it here. The notation is given by the following decomposition:

$$R_c(s) = 3\left[r_{c,0}(s) + r_{c,2}(s) + r_{c,4}(s)\right] + \ldots\,, \qquad (11)$$

which means that $r_{c,0}(s)$ corresponds to the massless approximation and $r_{c,2}(s)$ and $r_{c,4}(s)$ contains the mass corrections of order m^2 and m^4, respectively.

The charm quark mass is taken in the $\overline{\mathrm{MS}}$ scheme and the renormalization scale is set to $\mu^2 = s$. One observes that the higher order terms in α_s are all of comparable magnitude, and the convergence of the perturbative series is not manifest. Compared to the Born result, however, the corrections in α_s are small such that the stability of the full prediction is almost unaffected by the higher order terms.

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