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$\mathcal{O}(\alpha_s^3)$ conversion relation between $\overline{\rm MS}$ and Euclidean quark masses*

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We report on the analytical calculation of NNNLO ($\mathcal{O}(\alpha_s^3)$) conversion factor between the $\overline{\rm MS}$ quark mass and the one defined in the so-called "Regularization Invariant" scheme. The NNNLO contribution in the conversion factor turns out to be relatively large and comparable to the known NNLO term.

1. Introduction

Quark masses are fundamental parameters of the QCD Lagrangian. Nevertheless, their relation to measurable physical quantities is not direct: the masses depend on the renormalization scheme and, within a given one, on the renormalization scale μ.

In the realm of pQCD the most often used definition is based on the $\overline{\rm MS}$-scheme [1,2] which leads to the so-called short-distance $\overline{\rm MS}$ mass. Such a definition is of great convenience for dealing with mass-dependent inclusive physical observables dominated by short distances (for a review see [3]). Unfortunately, as their mass dependence is relatively weak the predictions are usually difficult to use for getting a precise information on quark masses.

To determine the absolute values of quark masses, one necessarily has to rely on the methods which incorporate the features of nonperturbative QCD. So far, the only two methods which are based on QCD from the first principles are QCD sum rules and lattice QCD (for recent discussions see e.g. [4–6,8–13]). Rather accurate determinations of the ratios of various quarks masses can be obtained within Chiral Perturbation Theory [14].

Lattice QCD provides a direct way to determine a quark mass from first principles. Unlike QCD sum rules it does not require model assumptions. It is possible to carry out the systematic improvement of lattice QCD so that all the discretization errors proportional to the lattice spacing are eliminated (a comprehensible review is given in [16]). The resulting quark mass is the (short distance) bare lattice quark mass. The matching of the lattice quark masses to those defined in a continuum perturbative scheme requires the calculation of the corresponding multiplicative renormalization constants. In the RI scheme [15] the renormalization conditions are applied to amputated Green functions in Landau gauge, setting them equal to their tree-level values. This allows the non-perturbative calculation of the renormalization constants. An alternative to the RI approach is the Schroedinger functional scheme (SF) which was used in [17,18]

Once the RI quark masses are determined from lattice calculations they should be related to the $\overline{\rm MS}$ mass by a corresponding conversion factor. By necessity this factor can be defined and, hence, computed only *perturbatively*. The conversion factor is now known at next-to-next-to-leading order (NNLO) from Ref. [19]. The NNLO contribution happens to be numerically significant. This makes mandatory to know the NNNLO $O(\alpha_s^3)$ term in the conversion factor.

In this work we report on the calculation of this term. It turns out that the size of the newly computed term is comparable to the previous one at a renormalization scale of 2 GeV — the typical scale currently used in the lattice calculations of the light quark masses. This means that perturbation theory can not be used for a precise conversion of the presently available RI quark masses to the $\overline{\rm MS}$ ones. A simple analysis shows that the

*Invited talk given by K.G.Ch. at the QCD 99 Euroconference, Montpellier, France 7-13th July 1999 convergency gets better if the scale is increased to, say, 3 GeV. Thus, once the lattice calculations produce the RI quark masses at this scale our formulas will allow accurate conversion to the $\overline{\rm MS}$ masses at the same scale.

2. Scheme dependence of quark mass

We start by considering the bare quark propagator (for simplicity we stick to the Landau gauge and, thus, do not explicitly display the gauge dependence)

$$F_0(q, \alpha_s^0, m_0) = (m_0 - \not{q} - \Sigma_0)^{-1}, \tag{1}$$

with the quark mass operator Σ_0 being conveniently decomposed into Lorentz invariant structures according to

$$\Sigma_0 = \not{q}\Sigma_V^0 + m_0 \Sigma_S^0. \tag{2}$$

Here m_0 and ψ_0 is the bare quark mass and field respectively: $a_s \equiv \frac{\alpha_s}{\pi} = \frac{g^2}{4\pi^2}$ and g is the bare QCD gauge coupling. To be precise we assume that (1) is dimensionally regulated by going to non-integer values of the space-time dimension $D = 4 - 2\epsilon$ [20,21]. The $\overline{\rm MS}$ renormalized counterpart of the Green function (1) reads

$$\begin{aligned} &F(q, \alpha_s, m, \mu) = (m - \not{q} - \Sigma)^{-1} \\ &= Z_2^{-1} F_0(q, \alpha_s^0, m_0)|_{m_0 = Z_m m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha \alpha_s}, \end{aligned} \tag{3}$$

where the renormalized quark field $\psi = Z_2^{-1/2}\psi_0$ and the 'Hooft mass parameter μ is a scale at which the renormalized quark mass is defined. The renormalization constants Z_2, Z_α and Z_m are series of the generic form ($? = 2, \alpha$ or m)

$$Z_? = 1 + \sum_{i>0} Z_?^{(i)} \frac{1}{\epsilon^i}, \quad Z_?^{(i)} = \sum_{j \geq i} Z_?^{(i,j)} \left(\frac{\alpha_s}{\pi}\right)^j. \tag{4}$$

Now let us consider the quark propagator renormalized according to a different subtraction procedure. Marking with a prime parameters of the second scheme one can write

$$\begin{aligned} &F(q, \alpha_s', m', \mu) = \frac{1}{m' - \not{q} - \Sigma'} \\ &= (Z_2')^{-1} F_0(q, \alpha_s^0, m_0)|_{m_0 = Z_m' m,\, \alpha_s^0 = \mu^\epsilon Z_\alpha' \alpha_s}, \end{aligned} \tag{5}$$

where without essential loss of generality we have set $\mu' = \mu$. The finiteness of the renormalized fields and parameters in both schemes implies that, within the framework of perturbation theory, the relation between them can be uniquely described as follows

$$m = C_m \cdot m', \quad \psi = \sqrt{C_2} \cdot \psi', \tag{6}$$

with the "conversion functions" being themselves *finite* series in α_s', i.e.

$$C_? \equiv 1 + \sum_{i>0} C_?^{(i)} \left(\frac{\alpha_s'}{\pi}\right)^i \tag{7}$$

for $? = m$ or 2.

Note that in general the coefficients $C_?^i$ may depend on the ratio m'/μ. If such a dependence is absent then the corresponding subtraction scheme is referred to as a "mass independent" one. In what follows we limit ourselves to considering the latter case. In addition, being only interested in conversion functions C_2 and C_m, we will assume that the function C_α has already been determined and, thus, will deal with the following representation of C_2 and C_m in terms of the $\overline{\rm MS}$ coupling constant α_s ($? = 2, m$)

$$C_? \equiv 1 + \sum_{i>0} C_?^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i \tag{8}$$

From eqs. ((1)) it is easy to see that

$$C_2 \cdot (1 + \Sigma_V) = 1 + \Sigma_V', \tag{9}$$

$$C_2 \cdot C_m \cdot (1 - \Sigma_S) = 1 - \Sigma_S'. \tag{10}$$

The renormalization conditions for the non-$\overline{\rm MS}$ scheme should then be used to provide the necessary information about the right hand side to calculate the conversion factors C_m and C_2 given the $\overline{\rm MS}$ renormalized Σ_V and Σ_S.

An example of a mass independent MOM scheme has recently been suggested in [15] under the name of RI ("Regularization Invariant") scheme. The corresponding versions of the general conversions formulas read [22] ($\ell = log(-\frac{q^2}{\mu^2})$)

$$C_2^{RI} = \left[1 + \Sigma_V + \frac{1}{2} \cdot \frac{\partial \Sigma_V(\ell)}{\partial \ell}\right]^{-1}_{q^2 = -\mu^2,\, m=0}, \tag{11}$$

$$C_m^{RI} = \left[\frac{1+\Sigma_V + \frac{1}{2}\cdot\frac{\partial \Sigma_V(\ell)}{\partial \ell}}{1-\Sigma_S} \right]_{q^2=-\mu^2\ m=0} . \quad (12)$$

3. Three Loop Conversion functions

We have analytically computed the functions Σ_V and Σ_S in the massless limit to order α_s^3. The calculation has been done with intensive use of computer algebra programs. In particular, we have used `QGRAF` [23] for the generation of diagrams, and `MINCER` [24] for their evaluation.

Our results for the conversion functions read (in the Landau gauge and in terms of fractions and Riemann's Zeta function and as functions of n_f):

$$\begin{aligned} C_2^{\rm RI} &= 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{517}{18} + 12\,\zeta_3 + \frac{5}{3}\,n_f \right] \\ &+ \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{1287283}{648} + \frac{14197}{12}\,\zeta_3 \right. \\ &\quad + \frac{79}{4}\,\zeta_4 - \frac{1165}{3}\,\zeta_5 + \frac{18014}{81}\,n_f \\ &\quad \left. - \frac{368}{9}\,\zeta_3\,n_f - \frac{1102}{243}\,n_f^2 \right], \end{aligned}$$

$$\begin{aligned} C_m^{\rm RI} &= 1 + \frac{\alpha_s}{4\pi} \left[-\frac{16}{3} \right] \\ &+ \left(\frac{\alpha_s}{4\pi}\right)^2 \left[-\frac{1990}{9} + \frac{152}{3}\,\zeta_3 + \frac{89}{9}\,n_f \right] \\ &+ \left(\frac{\alpha_s}{4\pi}\right)^3 \left[-\frac{6663911}{648} + \frac{408007}{108}\,\zeta_3 \right. \\ &\quad - \frac{2960}{9}\,\zeta_5 + \frac{236650}{243}\,n_f - \frac{4936}{27}\,\zeta_3\,n_f \\ &\quad \left. + \frac{80}{3}\,\zeta_4\,n_f - \frac{8918}{729}\,n_f^2 - \frac{32}{27}\,\zeta_3\,n_f^2 \right]. \end{aligned}$$

At a scale $\mu = 2$ GeV and $n_f = 4$, the numerical contributions of various orders are as follows (for simplicity we took $\alpha_s(2\,\mathrm{GeV})/\pi = .1$)

$$C_m^{RI} = 1. - 0.1333 - 0.0754 - 0.0495 \quad (13)$$

and

$$C_2^{RI} = 1. + 0. - 0.00477 - 0.00508. \quad (14)$$

One observes that the sizes of the NNLO and NNNLO contributions to C_m^{RI}, at this scale amount to about 7.5% and 5% respectively. This means that perturbation theory can not be used for a precise conversion of the RI quark masses to the $\overline{\rm MS}$ ones at the renormalization scale $\mu = 2$ GeV. The convergency can be improved if one icreases μ till, say, 3 GeV. Indeed, with this choice of μ the standard three-loop evolution gives $\alpha_s(3\ \mathrm{GeV}) = 0.262$ and eqs. (13,14) transform to

$$C_m^{RI} = 1. - 0.111 - 0.0526 - 0.0289 \quad (15)$$

and

$$C_2^{RI} = 1. - 0.00333 - 0.00296. \quad (16)$$

To conclude: the newly computed NNNLO corrections are numerically significant and should be taken into account when transforming the RI quark masses to the $\overline{\rm MS}$ ones.

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