BUTP–99/15
TTP99–33
hep-ph/9907509
July 1999

Short Distance Mass of a Heavy Quark at Order α_s^3

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The relation between the on-shell quark mass and the mass defined in the modified minimal subtraction scheme is computed up to order α_s^3. Implications for the numerical values of the top and bottom quark masses are discussed. We show that the new three-loop correction significantly reduces the theoretical uncertainty in the determination of the quark masses.

PACS numbers: 12.38.Bx 12.38.-t 14.65.Ha 14.65.Fy

In Quantum Chromodynamics (QCD) practical calculations are very often performed in the modified minimal subtraction ($\overline{\rm MS}$) scheme [1,2] leading to the definition of the so-called short-distance $\overline{\rm MS}$ mass. The $\overline{\rm MS}$ mass occupies a distinguished place among various mass definitions. First, it is a truly short distance mass not suffering from nonperturbative ambiguities. Second, the $\overline{\rm MS}$ mass proves to be extremely convenient in multi-loop calculations of mass-dependent inclusive physical observables dominated by short distances (for a review see [3]). On the other hand the experiments often provide masses which are tightly connected to the on-shell definition. Thus, conversion formulae are needed in order to make contact between theory and experiment. The two-loop relation between $\overline{\rm MS}$ and the on-shell definition of the quark mass has been obtained in [4]. Until recently the accuracy of this equation was enough for the practical applications. Meanwhile, however, new computations have become available which require the relation between the $\overline{\rm MS}$ and on-shell mass at $\mathcal{O}(\alpha_s^3)$ in order to perform a consistent analysis. The necessity of an accurate determination of the quark masses, especially those of the top and bottom ones, is demonstrated by the following two examples:

(*i*) The main goal of the future B physics experiments is the determination of the Cabibbo-Kobayashi-Maskawa matrix elements which will give deeper insight into the origin of CP violation and possibly also provides hints to new physics. In particular the precise measurement of V_{cb} is very promising. It is determined from semileptonic B meson decay rates. Thus it is desirable to know the bottom quark mass as accurately as possible as it enters already the Born result to the fifth power.

(*ii*) One of the primary goals of a future electron-positron linear collider (NLC) or muon collider (FMC) will be the precise determination of the top quark properties, especially its mass, M_t. In hadron colliders like the Fermilab TEVATRON or the Large Hadron Collider (LHC) the top quarks are reconstructed from the invariant mass of the W bosons and the bottom quarks. On the contrary in lepton colliders it is possible to determine the top quark mass from the line shape of the production cross section $\sigma(e^+e^- \to t\bar{t})$ close to the threshold. Simulation studies have shown that an experimental uncertainty of $100 - 200$ MeV in the top mass determination can be achieved [5]. Thus also from the theoretical side the ambiguities have to be controlled with the same precision. In particular in this context much attention has been devoted to the relation of the pole mass, M, to the $\overline{\rm MS}$ mass, m. Although the pole mass demonstrates a bad infra-red behaviour it is often convenient to use it in intermediate steps.

The connection between the $\overline{\rm MS}$ and on-shell mass is given by

$$m(\mu) = z_m(\mu)M \,, \tag{1}$$

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where z_m is finite and has an explicit dependence on the renormalization scale μ. In [6] the infra-red finiteness and the gauge invariance of M was proven. In [4] the perturbative expansion of z_m has been computed up to order α_s^2. The main purpose of this letter is the computation of z_m up to order α_s^3. Therefore three-loop corrections to the fermion propagator have to be considered. It is convenient to parameterize them in the following form

$$z_m(M) = 1 + a_M C_F z_m^F + a_M^2 \left[C_F^2 z_m^{FF} + C_F C_A z_m^{FA} + C_F T n_l z_m^{FL} + C_F T z_m^{FH} \right] + a_M^3\, z_m^{(3)} + \mathcal{O}\left(a_M^4\right) , \tag{2}$$

with $a_M = \alpha_s^{(n_f)}(M)/\pi$. n_l is the number of light (massless) quarks. In the following we will use $z_m = z_m(M)$.

The relation between the $\overline{\rm MS}$ and on-shell mass is obtained from the requirement that the inverse fermion propagator has a zero at the position of the on-shell mass. Thus in principle it is necessary to evaluate three-loop on-shell integrals in order to obtain the $\mathcal{O}(\alpha_s^3)$ term in the $\overline{\rm MS}$–on-shell mass relation. This is avoided by computing expansions of the quark self energy for small and large external momentum. After building the proper combinations needed for the relation between the $\overline{\rm MS}$ and on-shell mass a conformal mapping $q^2/M^2 = 4\omega/(1+\omega)^2$ is performed which maps the complex q^2-plane into the interior of the unit circle in the ω-plane. The relevant point $\omega = 1$ (corresponding to $q^2 = M^2$) is reached with the help of Padé approximation [7]. For further details we refer the reader to [8]. Note that this method has already been extremely successful in other applications [9].

Let us at this point discuss the one- and two-loop results in order to get some feeling about the quality of our procedure. Using our method the following numbers can be extracted

$$z_m^F = -1.0005(6)\,,\ z_m^{FF} = -0.49(2)\,,\ z_m^{FA} = -3.4(1)\,,\ z_m^{FL} = 1.566(4)\,,\ z_m^{FH} = -0.1553(2)\,, \tag{3}$$

where the error is obtained by doubling the spread of the different Padé approximants. The comparison with the exact result results [4] $\{-1, -0.51056, -3.33026, 1.56205, -0.15535\}$ shows very good agreement.

The results obtained at one- and two-loop order encourage to apply the same procedure also at order α_s^3. For the technical details we also refer to [8]. We just want to mention that in total we were able to evaluate 14 input terms for the Padé procedure.

We could apply the described procedure to each colour factor separately [8]. For most practical applications in QCD the number of flavours may be set to 3, i.e. $C_F = 4/3$ and $C_A = 3$, and $T = 1/2$ can be adopted. Adding in a first step the results for the moments and performing the Padé approximations afterwards leads us to

$$\frac{m(M)}{M} = 1 - 1.333 a_M + a_M^2\left[-14.33 + 1.041 n_l\right] + a_M^3\left[-202(5) + 27.3(7) n_l - 0.653 n_l^2\right] , \tag{4}$$

$$\frac{\mu_m}{M} = 1 - 1.333 a_M + a_M^2\left[-11.67 + 1.041 n_l\right] + a_M^3\left[-170(5) + 24.8(7) n_l - 0.653 n_l^2\right] , \tag{5}$$

$$\frac{M}{m(m)} = 1 + 1.333 a_m + a_m^2\left[13.44 - 1.041 n_l\right] + a_m^3\left[194(5) - 27.0(7) n_l + 0.653 n_l^2\right] , \tag{6}$$

with $a_m = \alpha_s^{(n_f)}(m)/\pi$. For simplicity $\mu = M$ and $\mu = m$ has been chosen in (4) and (6), respectively. In the above equations the exact values of the n_l^2 term [10] is displayed. Our method leads to 0.656(8) which is in very good agreement. This is a further justification for our approach.

The results for the values of $n_l = 0, \ldots, 5$ are summarized in Tab. I where also the two-loop coefficients are displayed. Actually the coefficients of the terms linear in n_l have been obtained by performing a fit to the three-loop results of Tab. I. The errors of about 2–3% for the three-loop results of (4)–(6) and Tab. I have again been obtained by doubling the spread of Padé approximants. On one side this is justified with the behaviour at $\mathcal{O}(\alpha_s^2)$ where the results of the first column in Tab. I are reproduced with an accuracy below 2%. On the other side the Padé approximants demonstrate more stability in the case where the moments are added and n_l is fixed afterwards than in the case where the Padé procedure is applied to the individual colour structures separately. We want to stress that the errors in Eqs. (4)–(6) are somewhat overestimated once a value for n_l is chosen. Thus for practical applications Tab. I should be used.

Let us next compare our results with various predictions which already exist in the literature obtained with the help of different optimization procedures. In Tab. II the results obtained for $M/m(m)$ using the fastest apparent convergence (FAC) [11] and the principle of minimal sensitivity (PMS) [12] are compared with ours. For $n_l = 2$ the discrepancy with our central value amounts to only 7%. It even reduces to 2% for $n_l = 5$, i.e. in the case of the top quark. The predictions obtained in the large-β_0 limit, where β_0 is the first coefficient of the QCD β function, are also shown in Tab. II. Excellent agreement below 1% is found for $n_l = 3$. It amounts to roughly 5% for $n_l = 4$ and 14% for $n_l = 5$.

In the remaining part of this letter we will discuss some important applications of the new term of $\mathcal{O}(\alpha_s^3)$ in the $\overline{\rm MS}$–on-shell relation.

Threshold phenomena are conveniently expressed in terms of the pole mass. To be specific let us consider the production of top quarks in e^+e^- collisions. The corresponding physical observables expressed in terms of M_t show in general a bad convergence behaviour. In the case of the total cross section the next-to-next-to-leading order corrections [13–15] partly exceed the next-to-leading ones. Furthermore the peak position which is the most striking feature of the total cross section and from which finally the mass value can be extracted depends strongly on the number of terms one includes into the analysis. The reason for this is that the pole mass is sensitive to long-distance effects which results in intrinsic uncertainties of order $\Lambda_{\rm QCD}$ [16].

Several strategies have been proposed to circumvent this problem [17,18,15]. They are based on the observation that the same kind of ambiguities also appear in the static quark potential, $V(r)$. In the combination $2M_t+V(r)$, however, the infra-red sensitivity drops out. Thus a definition of a short-distance mass extracted from threshold quantities should be possible. The relation of the new mass parameter to the pole mass is used in order to re-parameterize the threshold phenomena. On the other hand a relation of the new quark mass to the $\overline{\rm MS}$ mass must be established as it is commonly used for the parameterization of those quantities which are not related to the threshold. In order to do this consistently the three-loop relation between the $\overline{\rm MS}$ and the on-shell mass is needed.

In [17] the concept of the so-called potential mass, $m_{t,PS}$, has been introduced. Its connection to the pole mass is given by $m_{t,PS}(\mu_f) = M_t - \delta m_t(\mu_f)$ where $\delta m_t(\mu_f)$ is obtained from the static quark potential. In this way a subtracted potential, $V(r,\mu_f)$, is defined. The factorization scale μ_f has been introduced in order to extract the infra-red behaviour arising from the potential. In the combination $M_t - \delta m_t(\mu_f)$ it cancels against the one of M_t leading to an significant reduction of the long-distance uncertainties in $m_{t,PS}$ [17]. Thus it is promising to formulate the threshold problems in terms of $m_{t,PS}(\mu_f)$ and $V(r,\mu_f)$ instead of M_t and $V(r)$. In the final result the dependence on μ_f cancels. For the numerical analyses the value $\mu_f = 20$ GeV has been adopted in [14] as its upper bound is roughly given by $M_t C_F \alpha_s(\mu)$. $\delta m_t(\mu_f)$ is known up to order α_s^3 [17]. Thus we are now in the position to establish a relation between the two short-distance masses $m_{t,PS}(\mu_f)$ and $m_t(\mu)$ with the result:

$$m_{t,PS}(20\ {\rm GeV}) = (165.0 + 6.7 + 1.2 + 0.28)\ {\rm GeV}\,, \tag{7}$$

where the different terms represent the contributions of order α_s^0 to α_s^3. For the numerical values $m_t(m_t) = 165.0$ GeV and $\alpha_s^{(6)}(m_t(m_t)) = 0.1085$ have been used. Note that the error of the $\mathcal{O}(\alpha_s^3)$ coefficient in the $\overline{\rm MS}$–on-shell mass relation is negligible. The comparison of Eq. (7) with the analogous expansion for M_t,

$$M_t = (165.0 + 7.6 + 1.6 + 0.51)\ {\rm GeV}\,, \tag{8}$$

shows that the potential mass can be more accurately related to the $\overline{\rm MS}$ mass than M_t. The last term of the expansion in (7) is of the same order of magnitude as the error in the top quark mass determination at a NLC. Whereas in [14] this term has been taken as uncertainty in the mass relation the error reduces significantly after the knowledge of the $\mathcal{O}(\alpha_s^3)$ term of the $\overline{\rm MS}$–on-shell relation. This can be deduced from the well-behaved expansion in Eq. (7). The dominant error is now provided by the uncertainty in α_s.

A similar strategy has been proposed in [15]. There the so-called 1S mass, M_t^{1S}, has been defined as half the perturbative mass of a fictious toponium 1^3S_1 ground state which would exist if the top quark were stable. The philosophy is very similar as in the case of the potential mass. From the experiment the quantity M_t^{1S} is extracted. In [15] it has been shown that this is possible with an uncertainty of approximately 200 MeV. In a next step M_t^{1S} has to be related to the $\overline{\rm MS}$ mass $m_t(m_t)$. As the extraction of M_t^{1S} is based on a next-to-next-to-leading order formalism the $\mathcal{O}(\alpha_s^3)$ relation computed in this work is necessary.

In the practical calculation care has to be taken in connection to the expansion parameter which has to be used. Actually the so-called Υ-expansion has to be adopted. Details can be found in [19,20]. One finally arrives at the following relation between the $\overline{\rm MS}$ and $1S$ mass

$$m_t(m_t) = (175.00 - 7.60 - 0.97 - 0.14)\ {\rm GeV}\,, \tag{9}$$

where $M_t^{1S} = 175$ GeV and $\alpha_s^{(5)}(M_Z) = 0.118$ has been adopted. Using the large-β_0 results for the order α_s^3 term the last term reads -0.23 [15] which is off by more than 50% form the exact result. The conclusions which can be drawn from Eq. (9) are very similar to the ones stated above: the uncertainties due to unknown terms in the mass relations are negligible as compared to the error with which M_t^{1S} can be extracted from the experiment. The dominant uncertainty comes from the error in α_s which amounts for ± 0.003 to roughly 200 MeV [15] in Eq. (9).

Also the bottom quark mass can be extracted from quantities related to the quark threshold. Recently [21,20] a precise value for the bottom quark mass has been determined in the context of QCD sum rules. For example, in [20] the on-shell mass was eliminated in favour of the 1S mass in order to reduce the error. Once M_b^{1S} is determined, $m_b(m_b)$ can be found in analogy with the top quark case. As a result the values $M_b^{1S} = 4.71 \pm 0.03$ GeV and $m_b(m_b) = 4.2 \pm 0.06$ have been obtained. Following the procedure described in [20] one arrives at

$$m_b(m_b) = (4.71 - 0.40 - 0.11 - 0.03 \pm 0.03 \pm 0.04)\ \text{GeV}\,, \tag{10}$$

where the different terms correspond to different orders in the Υ-expansion. The first error is due to M_b^{1S} and the second one reflects the error in $\alpha_s^{(5)}(M_Z) = 0.118 \pm 0.004$ which is adopted from [20]. Due to the nice convergent behaviour in (10) the total error on $m_b(m_b)$ only contains these two sources which finally leads to

$$m_b(m_b) = 4.17 \pm 0.05\ \text{GeV}\,. \tag{11}$$

It is important to stress that taking into account of the newly computed $\mathcal{O}(\alpha_s^3)$ term in the $\overline{\text{MS}}$–on-shell relation is crucial for the reliable estimation of the errors in (11). Indeed, a deviation of the real value for $z_m^{(3)}$ from the large-β_0 estimation by, say, a factor of two, which one could not exclude a priori, would result to a systematic shift in $m_b(m_b)$ of around 100 MeV.

A somewhat different approach for the determination of both the charm and bottom mass has been followed in [22]. There the lower states in the heavy quarkonium spectrum were computed up to order α_s^4 and then used to extract the pole masses of b and c quarks. The transformation to the $\overline{\text{MS}}$ mass has been performed with the help of the two-loop relation [4]. However, to the order the quarkonium spectrum was computed it is more consistent to use the $\mathcal{O}(\alpha_s^3)$ relation provided in this paper. Thus taking over the error estimates from [22] the on-shell value for the bottom quark mass reads $M_b = 5.001^{+0.104}_{-0.066}$ GeV. It transforms to the following $\overline{\text{MS}}$ value

$$m_b(m_b) = 4.322^{+0.043}_{-0.028}\ \text{GeV}\,, \tag{12}$$

where the value $\alpha_s^{(5)}(M_Z) = 0.114$ has been used as in [22]. Compared to [22] the inclusion of the $\mathcal{O}(\alpha_s^3)$ terms leads to a shift in the central values of more than 100 MeV. In the case of the bottom quark this change is even larger than the errors presented in [22]. This demonstrates that a consistent treatment of the different orders in α_s is absolutely crucial.

To summarize: the computed value of the next-to-next-to-leading correction to the $\overline{\text{MS}}$–on-shell mass relation proves to be in a good agreement with an estimation based on the large-β_0 limit and has led to a significant reduction of the theoretical uncertainty in the determination of the quark masses.

We would like to thank A.H. Hoang for useful conversation and I. Bigi, J.H. Kühn and K. Melnikov for valuable suggestions and careful reading of the manuscript. This work was supported by DFG under Contract Ku 502/8-1 (*DFG-Forschergruppe "Quantenfeldtheorie, Computeralgebra und Monte-Carlo-Simulationen"*) and the *Schweizer Nationalfonds.*

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	$m(M)/M$		μ_m/M		$M/m(m)$	
n_l	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s^3)$
0	−14.33	−202(5)	−11.67	−170(5)	13.44	194(5)
1	−13.29	−176(4)	−10.62	−146(4)	12.40	168(4)
2	−12.25	−150(3)	−9.58	−123(3)	11.36	143(3)
3	−11.21	−126(3)	−8.54	−101(3)	10.32	119(3)
4	−10.17	−103(2)	−7.50	−81(2)	9.28	96(2)
5	−9.13	−82(2)	−6.46	−62(2)	8.24	75(2)

TABLE I. Dependence of $z_m^{(2)}$ and $z_m^{(3)}$ on n_l. The choice $\mu^2 = M^2$, respectively, $\mu^2 = m^2$ has been adopted. $z_m^{(2)}$ is defined as the sum of the terms inside the square brackets in Eq. (2).

n_l	this work	[23] (FAC)	[23] (PMS)	[10] (large-β_0)
2	143(3)	152.71	153.76	137.23
3	119(3)	124.10	124.89	118.95
4	96(2)	97.729	98.259	101.98
5	75(2)	73.616	73.903	86.318

TABLE II. Comparison of the results obtained in this paper with estimates based of FAC, PMS and the large-β_0 approximation for $M/m(m)$.