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Advanced method of solving recurrence relations for multi-loop Feynman integrals. ⋆

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Abstract

The systematic approach to solving the recurrence relations for multi-loop integrals is described. In particular, the criteria of their reducibility is suggested.

1 Introduction.

The integration by part method [1] is a very convinient and in most case unique opportunity to evaluate higher order perturbative corrections in quantum field theory. It provides with the relations connecting Feynman integrals with different powers of their denominators. In many cases it is possible to construct the recursive procedure which expresses an integral with given degrees of the denominators as a linear combination of a few so-called master integrals.

The construction of such procedure is a nontrivial problem even at two-loop level [2]. At three loops the case of vacuum integrals with one non-zero mass and various numbers of massless lines has been considered [3]. At four loops the recursive procedure was constructed only for some particular cases [4].

The recursive procedures mentioned above were constructed as the result of the "hand-work" combining of original relations. The total amount of this "hand-work" grows in very large extent with number of loops and starting from 4-loop level became unmanageable. Although there is some practical experience on that (unfortunately, not properly described in the literature), the whole approach is hard to be made automatic.

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From the other side, recently the systematic approach to solving the recurrence relations for multi-loop integrals was suggested [5], see also [6]. In this paper we describe the further progress in this direction.

2 Integral representation

The basic ideas of the approach is the following. Suppose we need to solve the relations for the objects $B(\underline{n}) \equiv B(n_1, ..., n_N)$:

$$R(I^-, I^+)B(\underline{n}) = 0, \tag{1}$$

$$I^-B(n_k) = B(n_k - \delta_{ki}), \quad I^+B(n_k) = n_i B(n_k + \delta_{ki}).$$

that is represent the arbitrary $B(\underline{n})$ as the sum of the irreducible objects $B(\underline{n}_k)$, where $\underline{n}_k, k = 1, ..., M$ are specific sets of index values (usually "zeros" and "ones"):

$$B(\underline{n}) = \sum_k c^k(\underline{n})B(\underline{n}_k), \tag{2}$$

Note that coefficient functions $c^k(\underline{n})$ are linear independent solutions of (1).(Proof: act by (1) on (2). If $B(\underline{n}_k)$ irreducible, than $R(I^-, I^+)c^k(\underline{n}) = 0$ for all k.)

By construction, the coefficient functions fulfil the initial conditions

$$c^i(\underline{n}_k) = \delta_{ik}. \tag{3}$$

If we have some set of solutions $f^{(k)}(\underline{n})$, we can construct a desirable set $c^k(\underline{n})$ as their linear combinations, which fulfil (3). Let us try the following complex-integral representation:

$$f^k(\underline{n}) = \int \frac{dx_1...dx_N}{x_1^{n_1}...x_N^{n_N}} g(\underline{x}). \tag{4}$$

Acting by (1) on (4) and using integration by part over x_i we got:

$$R(I^-, I^+)f^k(\underline{n}) = \int \frac{dx_1...dx_N}{x_1^{n_1}...x_N^{n_N}} R(x_i, \partial/\partial x_i)g(\underline{x}) + (\text{surface terms}). \tag{5}$$

(Due to the integration by part the order of operators in R changes to reverese one). Thus, choosing $R(x_i, \partial/\partial x_i)g(\underline{x}) = 0$ and removing surface terms by proper choice of integration contours we can fulfil the (1).

Suppose we are interesting in the solution which should be equal to zero if $n_i < 1$ for some i (for example, if the Feynman integral vanishes when degree of line number i is non-positive). In this case we can choose the integration contour for x_i as small circle around $x_i = 0$ and the integration will lead to the calculation of $(n_i - 1)^{\text{th}}$ Taylor coefficient in x_i. In the following we will call such n_i as "Taylor" type recurrence parameters.

Unfortunately, such "pure" case is not very practical. Instead of that, we ussually meet with "mixed" Taylor case, when Feynman integral vanishes if one shrinks some set of lines. Lets us explain, how this "mixed" case can be decomposed into the sum of "pure" cases.

3 Combinatorial decomposition

Let us consider first the simple example. Suppose we need to calculate $f_{n_1 n_2}$ (two-parameter recurrence problem), with additional condition $f_{n_1 n_2} = 0$ if $n_1 \leq 0$ and $n_2 \leq 0$. Let us define projectors I_i, O_i:

$$I_i f_{n_1 n_2} = (f_{n_1 n_2} \text{ if } n_i > 0 \text{ else } 0),\ \ O_i = 1 - I_i,\ \ i = 1, 2.$$

In these notations the condition on $f_{n_1 n_2}$ reads

$$O_1 O_2 f_{n_1 n_2} = 0. \tag{6}$$

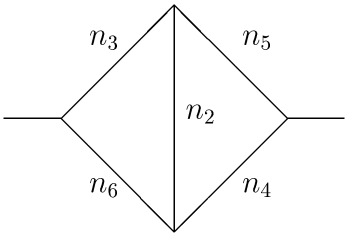
Then $f_{n_1 n_2}$ can be decomposed in the following way (in the second equality we omit $O_1 O_2$ contribution):

$$\begin{aligned} f_{n_1 n_2} &= (I_1 + O_1)(I_2 + O_2) f_{n_1 n_2} = (I_1 I_2 + I_1 O_2 + O_1 I_2) f_{n_1 n_2} = \\ &= (I_1 + I_2 - I_1 I_2) f_{n_1 n_2}. \end{aligned}$$

That is the original recurrence problem is represented as algebraic sum of the following sub-problems: the first when we neglecting all $f_{n_1 n_2}$ with n_1 non-positive, the second neglecting with n_2 non-positive, and the third when n_1 or n_2 is non-positive. Each of these sub-problems has "Taylor" type recurrence parameters, which significantly simplify the representation (4).

In the general case, we will have the list of conditions of (6) type; each item of this list corresponds to some case when the Feynman integral vanishes if one shrinks the given set of lines. To get the decomposition in general case, we need to calculate $\Pi_i (I_i + O_i)$, omit all monoms which include as sub-monom the item from the list, and in the rest ones make the substitution $O_i = 1 - I_i$. (Note, that one can obtain various versions of this decomposition by adding "zero" monoms from the list with arbitrary coefficients, but we found more practical the recipe described above).

As next example let us consider the 2-loop massless propagator type diagrams.



The "zero" list consists of $\{O_3O_5, O_3O_2, O_2O_5, O_4O_6, O_2O_6, O_4O_2, O_3O_6, O_4O_5\}$. The procedure described above leads to decomposition

$$f_{n_2 \ldots n_5} = (I_3I_4I_5I_6 + I_2I_3I_4 + I_2I_5I_6 - 2I_2I_3I_4I_5I_6) f_{n_2 \ldots n_5}. \tag{7}$$

The representation (4) in this example will read

$$f_{n_2 \ldots n_5} = \int \frac{dx_2 \ldots dx_6}{x_2^{n_2} \ldots x_6^{n_6}} P(x_1, \ldots, x_6)^{D/2-2}\,, \tag{8}$$

$$\begin{aligned} P = &(x_1 + x_2)(x_1x_2 - x_3x_4 - x_5x_6) + (x_3 + x_4)(x_3x_4 - x_1x_2 - x_5x_6) \\ &+ (x_5 + x_6)(x_5x_6 - x_1x_2 - x_3x_4) + x_1x_3x_6 + x_1x_4x_5 + x_2x_3x_5 + x_2x_4x_6, \end{aligned}$$

where x_1 corresponds to external line. Since the dependence on the external momentum is trivial, in the following we can set $x_1 = 1$ without loss of generality (the strict proof can be done by auxiliary integration over external momentum q^2 with weight $1/(q^2 - 1)$).

Let us now apply to (8) the decomposition (7). The first term in (7) implies the Couches integration over x_3, x_4, x_5, x_6, which leads to Taylor expansion in these variables. As the result we get a set of integrals of the following type:

$$\int \frac{dx_2}{x_2^{n_2}} [x_2(x_2 + 1)]^{D/2-2-c} \propto (-1)^{n_2 - c} \frac{(D/2 - 1)_{-c}(D/2 - 1)_{-c-n_2}}{(D - 2)_{-2c-n_2}}$$

The second term in (7) leads to Taylor expansion in x_2, x_3, x_4, the remaining integrals are:

$$\int \frac{dx_5 dx_6}{x_5^{n_5} x_6^{n_6}} [x_5x_6(1 - x_5 - x_6)]^{D/2-2-c} \propto \frac{(D/2 - 1)_{-n_5-c}(D/2 - 1)_{-n_6-c}}{(3D/2 - 3)_{-n_5-n_6-3c}}$$

The third term leads to similar contribution (with substitutions $3 \leftrightarrow 5, 4 \leftrightarrow 6$). Finally, the forth term leads to zero contribution (the polinom P vanish if $x_2 = x_3 = x_4 = x_5 = x_6 = 0$).

Let us consider this last sub-case more attentively. According to previous definitions, in the corresponding recurrence sub-problem we should omit any integral with $n_i \leq 0$ for some i, that is at least with one line (with label i) shrinked. The zero answer means that recurrence procedure re-express the integrals with all $n_i > 0$ (in particular with all $n_i = 1$) through the integrals with at least one line shrinked.

As the result, we get the necessary condition of reducibility: the given integral with $n_i > 0, i \in S$ can be expressed through more simple integrals (with some line shrinked) only if the Couches integration over the corresponding $x_i, i \in S$ in (4) leads to zero result:

$$\Pi_{i \in S} I_i f(\underline{n}) = 0. \tag{9}$$

Note, that in more complicated cases it may happen that the integrand does not vanish after Taylor expansion and the zero result can appear after remaining integration; we mean this final zero.

Moreover, it looks like (9) is at the same time the sufficient condition. The formal proof requires consideration of some pathological cases and still not completed. From the other hand, in practice we should construct the reduction procedure explicitly. In cases we met up to now (up to 3-loop propagator type massless integrals), if (9) takes place, it appeared to be possible. We will describe it in the expanded version of this paper.

4 Conclusion.

In this paper we shortly describe the systematic approach to solving the recurrence relations for multi-loop integrals. In particular, the criteria of their reducibility is suggested. We believe that this approach will allow to make automatic the solving procedure which help in the practical calculation of such integrals.

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