Recent results with heavy quarks and light lepton bound states[a]

Andrzej Czarnecki
Physics Department, Brookhaven National Laboratory,
Upton, NY 11973

Kirill Melnikov
Institut für Theoretische Teilchenphysik,
Universität Karlsruhe, D–76128 Karlsruhe, Germany

Alexander Yelkhovsky
Budker Institute for Nuclear Physics
Novosibirsk, 630090, Russia

Recent results on second order quantum corrections to semileptonic decays of b quarks and positronium spectroscopy are reviewed. Similarities between calculations in these seemingly different problems are demonstrated. Technical aspects of solving systems of recurrence relations, and application of dimensional regularization to bound states are explained.

1 Introduction

Precision measurements play a significant role in contemporary particle physics, spanning the energy scales starting with the Z-boson pole at the high end, down to scales of the order of the b and c quark masses, and further down to hydrogen, positronium, and muonium spectroscopy, and anomalous magnetic moments of leptons. The role of the precision measurements reflects the successes achieved by the Standard Model (SM) of electroweak and strong interactions. It has become clear in recent years that finding any signal of "New Physics," beyond the SM expectations, is a very difficult task.

Searching for hints about a more fundamental theory, which would replace the SM, but lacking devices necessary to make the next step up in energy, one resorts to a more precise analysis of the already available data and tries to increase the accuracy of the experiments at achievable energies. As a consequence, more and more accurate theoretical predictions are required practically at every point of the energy scale up to ~ 100 GeV.

Theoretical studies are often impeded by complications inherent to bound states or QCD uncertainties. Consider for example the inclusive decay width of the B-meson, $\Gamma(B \to X_c e\nu)$, used for the $|V_{cb}|$ determination. To calculate

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the decay rate in a model-independent way, one first has to clarify how to treat the non-perturbative bound state effects. This non-trivial issue is solved by invoking the Heavy Quark Expansion which permits a calculation of the inclusive semileptonic decay width as a systematic expansion [1] in powers of $\Lambda_{\rm QCD}/m$. The lowest order of this expansion corresponds to the free quark decay. There remains a technical difficulty of performing the standard perturbative calculations. Until recently, only one-loop accuracy was achievable for charged particle decays. Our approach to performing two-loop calculations is one of the subjects of this talk.

A complicated situation occurs also at the very low energies, where Coulombic bound states like hydrogen, muonium, or positronium are studied. The measurements in such systems are very precise. The theoretical predictions, however, are complicated by the essentially non-perturbative nature of bound state calculations. Fortunately, it turns out that the properties of the non-relativistic bound states, such as energy intervals and lifetimes, can be described in terms of series in powers and *logarithms* of the fine structure constant. Here a proper organization of the calculations is a challenging problem.

It is common in theoretical physics that problems can be solved if a small parameter is available. In case of the "standard" perturbative calculations the small parameter is usually the coupling constant. For the bound states, the smallness of a coupling constant is insufficient, because the very existence of the system can only be described in terms of an infinite number of Feynman diagrams. The situation is simplified in case of non-relativistic bound states, which encompass the light atoms and "leptonia" for which precision measurements are being performed. There, the relative velocity of the bound state components is small and can serve as an expansion parameter.

For the bound states, expanding around a non-relativistic limit is the only way known at present to arrive at precise results. But even in case of the more conventional perturbative calculations, Feynman integrals often are intractable and the only option to evaluate them is to *invent* a small parameter. It is often convenient to use some kinematic parameter, so that gauge invariance and other symmetries of the problem are preserved.

Construction of non-conventional expansions of Feynman graphs has recently helped to solve many outstanding problems in high precision theoretical studies. In what follows we are going to describe some examples of how such expansions can be constructed, and review our recent calculations devoted to the semileptonic decay width of the b quark and the positronium energy levels. We deliberately focus on the technical aspects of those calculations, to provide additional information and supplement our journal publications.

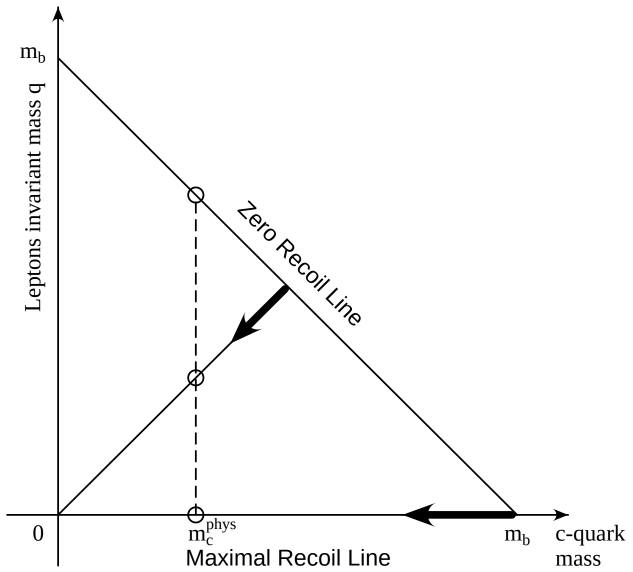

Figure 1: Status of the two-loop QCD corrections to the decay $b \to c +$ leptons. The dashed line denotes the physical region for the actual c quark. Points where the full corrections are known are circled. An analytical formula is known along the whole zero recoil line. [3,6] The other two points are found from expansions around the base points of the two arrows: at maximal recoil [5] and at the intersection with the diagonal. [4]

2 Inclusive semileptonic decay width of the b quark

Let us start by discussing a calculation of the second order QCD corrections, $\mathcal{O}(\alpha_s^2)$, to the semileptonic decay width of the b quark, $\Gamma(b \to c e \nu_e)$. At present a complete calculation is not feasible. However, one can calculate corrections at this order to the differential decay width $\mathrm{d}\Gamma/\mathrm{d}q^2_{e\nu}$ for special values of the leptonic invariant mass, $q^2_{e\nu} \equiv q^2$. Such calculations were performed at three points:

- $q^2 = (m_b - m_c)^2$ (zero recoil); [2,3]
- $q^2 = m_c^2$ (intermediate recoil); [4]
- $q^2 = 0$ (maximal recoil). [5]

These kinematical configurations are depicted in Fig. 1. Based on these 3 results, we performed a fit to interpolate the corrections for arbitrary q^2. From that fit, a correction to the inclusive decay rate, $\Gamma(b \to c e \nu_e)$, was estimated.

For kinematical configurations away from the zero recoil limit one should consider three different types of corrections: pure virtual corrections, single gluon radiation, and radiation of two gluons (at zero recoil only the first type is present). Our approach is to calculate these contributions separately and then sum them up in the final result.[b] Below we describe the essential ingredients of those calculations.

2.1 Virtual corrections

As an example of the virtual correction calculation we consider the intermediate recoil case, equating the invariant mass of the lepton system, $\sqrt{q^2}$, to the c-quark mass. If the c quark had exactly half the b quark mass, this configuration would coincide with the zero-recoil limit (see Fig. 1), in which case the virtual corrections are known exactly.[3,6] For the actual c and b masses these corrections can be calculated by expanding the amplitude in power series in $(m_b - 2m_c)/m_b$, that is around the known zero recoil case. An important aspect of this expansion is that it commutes with the virtual momenta integrations; it simply amounts to Taylor expanding the integrand.

After the expansion, all integrals we need are of the zero-recoil type. They can be divided up into the following categories (they are written here in a general form, with ω denoting the ratio of the final and the initial quark masses. For our intermediate recoil calculation we need only $\omega = 1/2$):

1. Planar with five propagators

$$P_5(a_1, a_2, a_3, a_4, a_5) = \pi^D \times \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{k_1^{2a_1} (k_1 - k_2)^{2a_2} k_2^{2a_3} \left(k_1^2 + 2k_1 p\right)^{a_4} \left(k_2^2 + 2k_2 p\omega\right)^{a_5}}. \tag{1}$$

2. Non-planar, first type, five propagators

$$N_P(a_1, a_2, a_3, a_4, a_5) = \pi^D \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{k_1^{2a_1} k_2^{2a_2}} \times \frac{1}{\left(k_1^2 + 2k_1 p\right)^{a_3} \left(k_2^2 + 2k_2 p\omega\right)^{a_4} \left[(k_1 + k_2)^2 + 2(k_1 + k_2)p\right]^{a_5}} \tag{2}$$

[b]An alternative approach to such calculations is based on the optical theorem; to get an n-loop result, one calculates the imaginary part of a certain $(n+1)$-loop Feynman integral. Such method was applied e.g. to calculate QED corrections to the muon decay rate.[7]

3. Non-planar, second type, five propagators

$$
\begin{aligned}
N'_P(a_1,a_2,a_3,a_4,a_5) &= \pi^D \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{k_1^{2a_1} k_2^{2a_2}} \\
&\times \frac{1}{(k_1^2+2k_1p)^{a_3}\,(k_2^2+2k_2p\omega)^{a_4}\,[(k_1+k_2)^2+2\omega(k_1+k_2)p]^{a_5}}
\end{aligned} \tag{3}
$$

4. Non-planar with six propagators

$$
\begin{aligned}
N_6(a_1,a_2,a_3,a_4,a_5,a_6) &= \pi^D \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{k_1^{2a_1} k_2^{2a_2}\,(k_1^2+2k_1p)^{a_3}} \\
\times \frac{1}{[(k_1+k_2)^2+2(k_1+k_2)p]^{a_4}\,[(k_1+k_2)^2+2\omega(k_1+k_2)p]^{a_5}\,(k_2^2+2\omega k_2p)^{a_6}}
\end{aligned} \tag{4}
$$

Taylor expansion of the amplitude may result in very many terms, differing by powers of the propagators, a_i. It is crucial to find a general algorithm of calculating them, so that an algebraic manipulation program can be employed. It is convenient to use recurrence relations derived using integration-by-parts,[8] to reduce an arbitrary integral to a small basis set of the so-called master integrals.

Let us note at this point that solving recurrence relations is still more of an "art" than "science." It would be very desirable to have a better understanding of the mathematical structure of the systems of recurrence relations; a very promising approach to this problem is being developed by P. Baikov.[9]

Here, we would like to demonstrate our approach to solving such recurrence relations with the example of the planar integral $P_5(\{a_i\})$. First, a system of relations is derived in the usual way[8] by equating integrals of total derivatives to zero. One finds

$$M_1 = D - 2a_1 - a_2 - a_4 + a_2\mathbf{2}^+(\mathbf{3}^- - \mathbf{1}^-) - a_4\mathbf{4}^+\mathbf{1}^-$$

$$M_2 = D - a_1 - 2a_2 - a_4 + a_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-) + a_4\mathbf{4}^+\left(\frac{\omega-1}{\omega}\mathbf{3}^- - \mathbf{2}^- + \frac{1}{\omega}\mathbf{5}^-\right)$$

$$M_3 = D - a_1 - a_2 - 2a_4 - a_1\mathbf{1}^+\mathbf{4}^- + a_2\mathbf{2}^+\left(\frac{\omega-1}{\omega}\mathbf{3}^- - \mathbf{4}^- + \frac{\mathbf{5}^-}{\omega}\right) + 2a_4\mathbf{4}^+$$

$$M_4 = D - 2a_2 - a_3 - a_5 + a_3\mathbf{3}^+(\mathbf{1}^- - \mathbf{2}^-) + a_5\mathbf{5}^+\left((1-\omega)\mathbf{1}^- - \mathbf{2}^- + \omega\mathbf{4}^-\right)$$

$$M_5 = 2D - 2a_1 - 2a_2 - 2a_3 - a_4 - a_5 - a_4\mathbf{4}^+\mathbf{1}^- - a_5\mathbf{5}^+\mathbf{3}^-$$

$$M_6 = D - a_2 - a_3 - 2a_5 + a_2\mathbf{2}^+\left((1-\omega)\mathbf{1}^- + \omega\mathbf{4}^- - \mathbf{5}^-\right)$$
$$-a_3\mathbf{3}^+\mathbf{5}^- + 2\omega^2 a_5\mathbf{5}^+$$

$$M_7 = \left(\frac{3D}{2} - \sum_{i=1}^{5} a_i\right)(\mathbf{1}^- - \mathbf{4}^-) + 2a_4\mathbf{4}^+\mathbf{1}^- + a_5\mathbf{5}^+\omega(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-)$$

$$M_8 = \frac{1}{\omega}\left(\frac{3D}{2} - \sum_{i=1}^{5} a_i\right)(\mathbf{3}^- - \mathbf{5}^-) + a_4\mathbf{4}^+(\mathbf{1}^- - \mathbf{2}^- + \mathbf{3}^-) + 2\omega a_5\mathbf{5}^+\mathbf{3}^- \quad (5)$$

The recursive algorithm proceeds in three steps:

1. the exponents of the massive propagators ($a_{4,5}$) are reduced to 1;

2. the exponent of the massless propagator containing both integration impulses (a_2) is reduced to 1;

3. the exponents $a_{1,3}$ are reduced.

The first step is simple. We assume that a_4 and a_5 are positive; otherwise the integral can be done by a sequence of one-loop integrations. We use the relation M_6. The term $2\omega^2 a_5\mathbf{5}^+$ involves an integral with a_5 higher than any other term in this relation. Therefore we can use M_6 to express $P_5(\ldots, a_5+1)$ by $P_5(\ldots, a_5)$. An analogous manipulation is done using M_3 to reduce a_4. Eventually we obtain $a_4, a_5 \leq 1$.

The next step is accomplished as follows. We first would like to eliminate all operators that contain $\mathbf{4}^+$ and $\mathbf{5}^+$ from the system (5). There are eight such operators $\mathbf{4}^+$, $\mathbf{4}^+\mathbf{1}^-$, $\mathbf{4}^+\mathbf{2}^-$, $\mathbf{4}^+\mathbf{3}^-$, and $\mathbf{5}^+$, $\mathbf{5}^+\mathbf{1}^-$, $\mathbf{5}^+\mathbf{2}^-$, $\mathbf{5}^+\mathbf{3}^-$. An important point is that this number is equal to the number of the recurrence relations and we can therefore solve the recurrence relations for these operators.

We denote the resulting 8 equations by $T_{4,41,42,43,5,51,52,53}$. The notations are such that for example T_{41} means an equation which expresses the operator $\mathbf{4}^+\mathbf{1}^-$ through other operators. These equations allow us to obtain new recurrence relations using $a_4(T_{41} - \mathbf{1}^-T_4) = 0$ and similar equations. In particular, we use a relation $a_4(T_{43} - \mathbf{3}^-T_4) = 0$ to lower the value of a_2 to 1.

After this has been achieved, we eliminate all operators which contain $\mathbf{2}^+$. At this stage, the problem is essentially two parametric (i.e. we have to move in the $a_1 - a_3$ plane). Finally, we are able to reduce any $P_5(\{a_i\})$ integral to four integrals $P_5(1,1,1,1,1)$, $P_5(0,1,1,1,1)$, $P_5(-1,1,0,1,1)$, and $P_5(0,1,0,1,1)$

and a number of trivial integrals. The four master integrals are

$$
\begin{aligned}
P_5(1,1,1,1,1) &= -\frac{\pi^2}{3\epsilon} - \frac{2\pi^2}{3} - \frac{5\zeta(3)}{2} - \pi^2 \ln 2 \\
P_5(0,1,1,1,1) &= \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left(\frac{5}{2} + 2\ln 2\right) + \frac{19}{2} + 10\ln 2 - \frac{5}{12}\pi^2 \\
P_5(-1,1,0,1,1) &= \frac{19}{16\epsilon^2} + \frac{1}{\epsilon}\left(\frac{311}{96} + \frac{3\ln 2}{4}\right) + \frac{1225}{192} + \frac{79}{24}\ln 2 - \frac{\ln^2 2}{2} + \frac{15\pi^2}{32} \\
P_5(0,1,0,1,1) &= -\frac{5}{8\epsilon^2} - \frac{1}{\epsilon}\left(\frac{29}{16} + \frac{\ln 2}{2}\right) - \frac{107}{32} - \frac{13}{4}\ln 2 + \ln^2 2 - \frac{11\pi^2}{48}.
\end{aligned}
$$

The above approach to solving recurrence relations is quite general. One sees that such solution cannot be constructed in a completely universal way. However, our approach allows to go through a number of steps, constructing a solution without much effort and it has proved useful in various cases. In particular, it works for the "hard" threshold integrals, necessary for the calculations described in Section 3.3.

2.2 Emission of two gluons in semileptonic b decays

For simplicity, we will discuss here the case of the maximal recoil, $q^2 = 0$. The intermediate recoil case is treated in an analogous way. A more detailed discussion of this issue is given in [10]. Here the most essential features are summarized.

The basic observation in this part of the calculation is that the energy losses caused by radiation off a non-relativistic particle are small. This is evident in the Coulomb gauge, since the interaction of magnetic photons with charged particles is proportional to the velocity of the latter. Therefore, the physical properties of the real radiation provide a hint how an expansion should be organized. Considering a decay $b \to c + e + \nu_e + g_1 + g_2$ in the case $q^2 \equiv (e + \nu_e)^2 = 0$, we intend to calculate it expanding in $\delta = (m_b - m_c)/m_b$. For brevity we will refer to the e, ν system as to the W boson. Taking into account phase space constraints, it is easy to establish that the spatial momentum of the c quark, as well as the four-momenta of the W, g_1 and g_2 are of the order δ. Then, a natural expansion of the propagators emerges: all propagators of the virtual particles can be expanded around the *static limit*, which essentially simplifies the calculation. Clearly, such an expansion will generate a large number of terms, but this can be handled using symbolic manipulation programs.

The last non-trivial question is the integration of the obtained expression over the four particle phase space. Given that the remaining propagators are static and the W boson does not participate in the interaction, it is clear that the integration over two-particle phase (sub)space (W, c) can be easily performed. The remaining part of the calculation requires an integration over the three particle phase space – two gluons and the "particle" composed of W and c. A systematic expansion of this object is accomplished by an appropriate choice of variables, described in detail in [10].

2.3 One gluon radiation and one virtual correction

Single gluon emission in the presence of a virtual loop is, in principle, possible to compute exactly. However, it is very inconvenient, e.g. because of necessity of integrating four-point functions over a three particle phase space.

We have adopted a different approach, based again on an expansion of such amplitudes around the static limit ($m_b = m_c$). In contrast to the virtual corrections and double gluon emission, in the present case the expansion is non-trivial. Let us consider the following diagram: first a virtual gluon is exchanged between c and b quarks, and then the c quark emits a real gluon. An essential point is that before the real gluon emission the c quark is *off mass shell*, which provides an infra-red regularization for the virtual loop. Expanding the one-loop amplitude naively, i.e. putting the "intermediate" c-quark line loop on shell, we change the infrared properties of the diagram and divergences will result in singularities in $1/(D-4)$ (we use dimensional regularization). To avoid these spurious singularities the expansion should be "corrected"; in this case the so-called eikonal expansion [11,12] is used. In the context of the two loop corrections to fermion decays this expansion is described in detail in Ref. [10].

2.4 General remarks on semileptonic heavy fermion decays

Let us summarize the technical experience we have gained by studying semileptonic b and t decays to second order in the strong coupling constant.

First, we note that the problem was made tractable by inventing a "small" parameter. For the top quark decays [13] the result is obtained if we take the "small" parameter close to unity. This clearly pushes the applicability of the method to its limits. However, even in this extreme case reasonable numerical estimates can be obtained.

Second, because we consider processes with the on–shell external particles, one has to be careful in analyzing contributing regions of integration momenta. In particular, one can encounter non-analytical dependence on the expansion parameter. In many cases the relevant momentum regions can be found by considering the corresponding effective theories.

Third, it is possible to construct an expansion of the amplitudes with real radiation of massless particles starting from the non-relativistic limit. It is likely that this construction can be applied beyond the semileptonic decay width, for example in a perturbative calculation of the cross section of such process as $e^+e^- \to \mu^+\mu^-\gamma\gamma$ close to the two muon threshold, or even for higher energies if one takes into account sufficiently many terms of such an expansion.

3 Bound States in QED

Consistent description of bound states is one of the most complicated problems in Quantum Field Theory. The problem is simplified if the bound state components are non-relativistic. This is the case, for example, in QCD in Υ or ψ meson families, or in QED in light atoms and in leptonic bound states like positronium or muonium.

The difficulty in the study of bound states is their essentially non–perturbative nature, as we discussed in Section 1. In the non-relativistic case the situation is simplified: to first approximation one can describe the bound state using a Schrödinger wave function. This description can subsequently be refined by means of perturbation theory, using the relative velocity v of the components as an expansion parameter.

This kind of expansions can be constructed both for QED and QCD. In what follows, we limit ourselves to the QED where a low-energy effective theory approach is well established, and is known as Non-Relativistic Quantum Electrodynamics (NRQED).[14]

3.1 NRQED

Very roughly speaking, the idea underlying NRQED consists in separating the soft scale $\sim m\alpha$, which characterizes the bound state, from the hard one $\sim m$, and "integrating out" the latter. The interactions at the soft scale can be described by Quantum Mechanics, i.e. employing a non-relativistic approximation. The physics at hard scales is governed by the relativistic QFT. However, when we are interested in processes which occur at small momenta and energies, the contribution of hard scales can be accounted for by introducing effective local operators and computing their coefficients in the effective Hamiltonian.

In the lowest order of this construction we describe the system by a bound state wave function. For positronium it is found from the Schrödinger equation with the "leading order" Hamiltonian (m is the electron mass, r is the distance between the electron and the positron):

$$H_{\rm LO} = \frac{\mathbf{p}^2}{m} - \frac{\alpha}{r}. \tag{6}$$

In a 3-dimensional space the ground state wave function (in momentum representation) is

$$\phi(p) = \sqrt{\frac{\pi \alpha m}{2}} \frac{2m^2\alpha^2}{(\mathbf{p}^2 - mE)^2}, \qquad E = -\frac{m\alpha^2}{4}. \tag{7}$$

The difference between the full Hamiltonian and Eq. (6) can be treated as a perturbation. The difficulties of calculating in NRQED are caused by divergences in the matrix elements of the subleading operators. In the original formulation [14] the ultraviolet divergences were dealt with by subtracting counterterms defined with help of diagrams evaluated on-shell and at threshold. Those diagrams, in turn, contain infrared divergences, which are regularized by introducing a small mass λ for the photon.

Since that pioneering work it has become clear that working with a massive photon leads to technical difficulties, particularly in evaluating the so-called hard corrections, arising at scales $\sim m$. Indeed, the hard corrections remain unknown (at two-loops) for such processes as para or orthopositronium decays, while the soft scale effects have already been evaluated. The reason for this is that the two-loop diagrams are difficult to evaluate when more than one mass scale is present. Although the only relevant scale in hard corrections for positronium physics is the electron mass, regularization with a photon mass introduces a second scale λ. The hard corrections can, however, be computed in many configurations, if instead of introducing λ one uses dimensional regularization.

It has been shown [15] that dimensional regularization permits an exact separation of effects arising at various characteristic energy scales. Using that method, the complete energy spectrum of Ps has been reproduced [16] to order $m\alpha^5$. Further, we have computed analytically $m\alpha^6$ corrections to the hyperfine splitting (HFS) of the Ps ground state. [17] More recently, [18] we generalized that result to all S states and computed also their spin independent shift at $\mathcal{O}(m\alpha^6)$. Here we would like to discuss some features of the NRQED in dimensional regularization, which we found useful in those calculations.

3.2 Quantum Mechanics in d dimensions

The first point we would like to discuss is a construction of the lowest order approximation. We consider the Schrödinger equation in a d-dimensional momentum space,

$$\phi(p) = \frac{4\pi\alpha m}{\mathbf{p}^2 - mE} \int \frac{\mathrm{d}^\mathrm{d}\mathbf{k}}{(2\pi)^d} \frac{\phi(k)}{(\mathbf{p}-\mathbf{k})^2}. \tag{8}$$

The solution of this equation for arbitrary d is not known. This complicates the calculations which involve the wave function. However, since in the configuration space all divergences are located at $r = 0$, it is possible to extract divergent contributions in the form of $\psi^2(0)/\epsilon$. Thus, the unknown overall factor $\psi^2(0)$ will be present in the intermediate results. In the final result the singularities $1/\epsilon$ cancel and we can replace $\psi(0)$ by its 3-dimensional value.

In rare cases we need, in addition to the value of the configuration space wave function at the origin, also the behavior of its first derivative. This, however, is related to $\psi(0)$ by the Schrödinger equation. To demonstrate this, we consider the Schrödinger equation in the configuration space. Since for S states $\psi(r)$ has no angular dependence, we have

$$\left(-\frac{\nabla^2}{m} + V(r)\right)\psi(r) = -\frac{1}{m}\left(\partial_r^2 + \frac{d-1}{r}\partial_r - mV(r)\right)\psi(r) = 0, \tag{9}$$

where $V(r)$ is the Coulomb potential in d dimensions,

$$V(r) = -\frac{\alpha\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}-1}r^{d-2}}. \tag{10}$$

We assume that the wave function near the origin behaves like $\psi(r) \simeq \psi(0)(1-cr^\beta)$, and find c and β by substituting this form into Eq. (9):

$$\psi(r\to 0) \simeq \psi(0)\left(1 - \frac{m\alpha\Gamma\left(\frac{1}{2}-\epsilon\right)}{2\pi^{\frac{1}{2}-\epsilon}(1+2\epsilon)}r^{1+2\epsilon}\right). \tag{11}$$

We now would like to demonstrate how divergent integrals are treated in practice, using two examples from our hyperfine splitting calculations. First, we consider a logarithmic divergence arising from tree-level Coulomb photon exchange,[18]

$$\Delta_C E_{\rm hfs} = -\frac{\pi\alpha}{dm^4}\left\langle\frac{(\mathbf{p}'\mathbf{q})(\mathbf{q}\mathbf{p})}{\mathbf{q}^2}\right\rangle. \tag{12}$$

In Eq. (12) the matrix element is to be calculated over the ground state wave function in d dimensions:

$$\langle f(\mathbf{p},\mathbf{p}')\rangle \equiv \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{\mathrm{d}^d p'}{(2\pi)^d}\phi(p)\phi(p')f(\mathbf{p},\mathbf{p}').$$

Although the integrand does not look complicated, the difficulty is that the exact form of the wave function $\psi(r)$ in d dimensions is not known. We use the fact that it satisfies the d-dimensional Schrödinger equation, Eq. (8). Using that equation we rewrite the integral in Eq. (12) as

$$\left\langle \frac{(\mathbf{p}'\mathbf{q})(\mathbf{q}\mathbf{p})}{\mathbf{q}^2} \right\rangle_{\mathbf{p},\mathbf{p}'} = \left\langle \frac{(4\pi\alpha m)^2(\mathbf{p}'\mathbf{q})(\mathbf{q}\mathbf{p})}{(\mathbf{p}^2 - mE)(\mathbf{p}-\mathbf{k})^2\mathbf{q}^2(\mathbf{p}'^2 - mE)(\mathbf{p}'-\mathbf{k}')^2} \right\rangle_{\mathbf{k},\mathbf{k}'}, \tag{13}$$

where the integration over $\mathbf{p}$, $\mathbf{p}'$, as well as $\mathbf{k}$, $\mathbf{k}'$, in the last expression is understood. The integral over $\mathbf{p}$ and $\mathbf{p}'$ receives a divergent contribution only from the region where $\mathbf{p}$ and $\mathbf{p}'$ become simultaneously infinite. Therefore, a single subtraction is sufficient to make this integral finite. It is convenient to subtract from (13) the following expression:

$$\left\langle \frac{(4\pi\alpha m)^2(\mathbf{p}'\mathbf{q})(\mathbf{q}\mathbf{p})}{(\mathbf{p}^2 - mE)^2\mathbf{q}^2(\mathbf{p}'^2 - mE)^2} \right\rangle_{\mathbf{k},\mathbf{k}'}. \tag{14}$$

After the subtraction is done, two nice features emerge. In Eq. (14) the integration over $\mathbf{k}, \mathbf{k}'$ factorizes and leads to $\psi^2(0)$ times a two-loop integral, which can be easily calculated for arbitrary d. On the other hand, the difference between the last integral in Eq. (13) and the integral in Eq. (14) is finite and can be calculated for $d = 3$ using the explicit form of the wave function, Eq. (7).

We note that the counterterm (14) is constructed in such a way that the difference mentioned above vanishes for the ground state. This can be easily seen by integrating over $\mathbf{k}, \mathbf{k}'$ in Eq. (14) and noticing that the $\mathbf{p}, \mathbf{p}'$-dependent terms in the denominator of Eq. (14) coincide (up to a normalization factor) with the three-dimensional ground state wave functions in the momentum representation.

Finally, for the Coulomb exchange contribution one finds

$$\Delta_C E_{\mathrm{hfs}} = \frac{\pi\alpha^3}{24m^2}\psi^2(0)\left(\frac{1}{\epsilon} - 4\ln(m\alpha) - \frac{1}{3}\right), \tag{15}$$

Our second example is a linearly divergent integral arising from the tree-level magnetic photon exchange,[18]

$$\begin{aligned}\langle \mathbf{p}^2 \rangle &= \psi(0)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\mathbf{p}^2\phi(p) \\ &= m\psi(0)\int \frac{\mathrm{d}^d p}{(2\pi)^d}\left(E\phi(p) + \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{4\pi\alpha}{(\mathbf{p}-\mathbf{k})^2}\phi(k)\right).\end{aligned} \tag{16}$$

Shifting the integration variable $\mathbf{p} \to \mathbf{p} + \mathbf{k}$ we find that the $\mathbf{p}$-integral in the last term is scale-less. In dimensional regularization such integrals vanish. The first term in Eq. (16) is finite in three dimensions. We obtain

$$\langle \mathbf{p}^2 \rangle = mE\psi^2(0). \tag{17}$$

3.3 Hard scale contributions

Perhaps the main advantage of using dimensional regularization in bound state calculations is the possibility of computing the hard corrections analytically.

The contributions of hard scales arise from virtual momenta of order of the electron mass. They can be calculated by considering the on–shell e^+e^- scattering amplitude exactly at the threshold, i.e. for zero relative velocity of the incoming electron and positron. Since the amplitude is independent of the incoming spatial momenta, it gives rise to four-fermion operators in the low-scale Lagrangian or, equivalently, to the $\delta(\boldsymbol{r})$ terms in the effective quantum mechanical Hamiltonian.

Technically, this calculation is similar to the derivation of the matching coefficient of the vector quark-antiquark current in QCD and its NRQCD counterpart. [19,20]

An arbitrary Feynman integral which contributes to the hard scale part of the calculation can be written as

$$I(a_1,...a_9) = \int \frac{\mathrm{d}^D k_1}{(2\pi)^D}\frac{\mathrm{d}^D k_2}{(2\pi)^D}\frac{1}{S_1^{a_1}S_2^{a_2}S_3^{a_3}S_4^{a_4}S_5^{a_5}S_6^{a_6}S_7^{a_7}S_8^{a_8}S_9^{a_9}}, \tag{18}$$

where

$$\begin{aligned}
&S_1 = k_1^2, \qquad S_2 = k_2^2, \qquad S_3 = (k_1-k_2)^2, \qquad S_4 = k_1^2 + 2pk_1,\\
&S_5 = k_2^2 + 2pk_2, \qquad S_6 = k_1^2 - 2pk_1, \qquad S_7 = k_2^2 - 2pk_2,\\
&S_8 = (k_1-k_2)^2 + 2p(k_1-k_2), \qquad S_9 = (k_1-k_2)^2 - 2p(k_1-k_2),
\end{aligned} \tag{19}$$

and $a_1, \ldots, a_9$ are integers. In practice we find diagrams with only at most 6 different propagators, so that at least 3 exponents a_i are zero.

A method for computing these integrals is analogous to the 2-loop virtual corrections to quark decays. We have described our algorithm in Section 2.1.

4 Conclusions

In this talk we have presented technical aspects of our recent calculations of two-loop corrections to semileptonic quark decays and positronium properties. The methods developed in the context of quark decays, in particular the algorithm for solving systems of recurrence relations, have found applications in positronium physics. We hope that further progress can be made in such atomic physics calculations using the tools developed for high energy problems. In particular, applying dimensional regularization should facilitate so far intractable calculations, such as two-loop corrections to positronium decays.

1. I. Bigi, M. Shifman, and N. Uraltsev, *Ann. Rev. Nucl. Part. Sci.* **47**, 591 (1997).
2. A. Czarnecki, *Phys. Rev. Lett.* **76**, 4124 (1996).
3. A. Czarnecki and K. Melnikov, *Nucl. Phys.* **B505**, 65 (1997).
4. A. Czarnecki and K. Melnikov, *Phys. Rev.* **D59**, 014036 (1999).
5. A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* **78**, 3630 (1997).
6. J. Franzkowski and J. B. Tausk, *Eur. Phys. J.* **C5**, 517 (1998).
7. T. van Ritbergen and R. G. Stuart, *Phys. Rev. Lett.* **82**, 488 (1999).
8. K. G. Chetyrkin and F. Tkachov, *Nucl. Phys.* **B192**, 159 (1981).
9. P. A. Baikov, *Phys. Lett.* **B385**, 404 (1996).
10. A. Czarnecki and K. Melnikov, *Phys. Rev.* **D56**, 7216 (1997).
11. A. Czarnecki and V. A. Smirnov, *Phys. Lett.* **B394**, 211 (1997).
12. V. A. Smirnov, *Phys. Lett.* **B394**, 205 (1997).
13. A. Czarnecki and K. Melnikov, hep-ph/9806244, in press in *Nucl. Phys. B*.
14. W. E. Caswell and G. P. Lepage, *Phys. Lett.* **167B**, 437 (1986).
15. A. Pineda and J. Soto, *Nucl. Phys. Proc. Suppl.* **64**, 428 (1998).
16. A. Pineda and J. Soto, *Phys. Rev.* **D59**, 016005 (1999).
17. A. Czarnecki, K. Melnikov, and A. Yelkhovsky, *Phys. Rev. Lett.* **82**, 311 (1999).
18. A. Czarnecki, K. Melnikov, and A. Yelkhovsky, hep-ph/9901394, in press in *Phys. Rev. A*.
19. A. Czarnecki and K. Melnikov, *Phys. Rev. Lett.* **80**, 2531 (1998).
20. M. Beneke, A. Signer, and V. A. Smirnov, *Phys. Rev. Lett.* **80**, 2535 (1998).