α^2 corrections to parapositronium decay

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Two-loop QED corrections to the decay rate of parapositronium (p-Ps) into two photons are presented. We find $\Gamma(\text{p-Ps} \to \gamma\gamma) = 7989.50(2)\,\mu\text{s}^{-1}$. The non-logarithmic $\mathcal{O}(\alpha^2)$ corrections turn out to be small, contrary to some earlier estimates. We speculate that the so-called "orthopositronium lifetime puzzle" will not likely be solved by large QED corrections.

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Positronium (Ps), the simplest known atom, is an ideal system to test quantum electrodynamics (QED) of bound states. The spectrum and lifetimes of Ps states are, at least in principle, calculable within QED with very high accuracy. Hadronic effects, which in other atoms limit the attainable theoretical precision, are suppressed by the small ratio of electron and hadron masses.

The lifetimes of the triplet and singlet ground states (respectively orthopositronium and parapositronium) have been subjected to very precise theoretical and experimental studies. Theoretical predictions for parapositronium (p-Ps) and orthopositronium (o-Ps) decay rates into 2 and 3 photons respectively, can be expressed as expansions in the fine structure constant α:

$$\Gamma^{\text{theory}}_{\text{p-Ps}} = \Gamma^{(0)}_p \left[1 - \left(5 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + 2\alpha^2 \ln\frac{1}{\alpha} + B_p\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\frac{1}{\alpha} + \dots \right] \tag{1}$$

$$\Gamma^{\text{theory}}_{\text{o-Ps}} = \Gamma^{(0)}_o \left[1 - 10.286\,606(10)\frac{\alpha}{\pi} - \frac{\alpha^2}{3}\ln\frac{1}{\alpha} + B_o\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\frac{1}{\alpha} + \dots \right] \tag{2}$$

where

$$\Gamma^{(0)}_p = \frac{m\alpha^5}{2}, \qquad \Gamma^{(0)}_o = \frac{2(\pi^2-9)m\alpha^6}{9\pi} \tag{3}$$

are the lowest order decay widths of the p-Ps and o-Ps, respectively, and the ellipses in eqs. (1,2) denote unknown higher order terms which we will neglect in our analysis. Corrections of $\mathcal{O}(\alpha)$ were calculated in [1] for p-Ps. For o-Ps the most accurate result was obtained in [2], where references to earlier works can be found. The logarithmic two-loop correction was found in [3] for o-Ps and in [4] for p-Ps. The leading logarithmic correction at three loops was computed in [5]. Some partial results on the $\mathcal{O}(\alpha^2)$ corrections for both p-Ps and o-Ps can be found in [2,6–9], but complete values of $B_{p,o}$ have not been obtained so far.

Using eqs.(1,2,3), one obtains the following theoretical predictions for the lifetimes

$$\Gamma^{\text{theory}}_{\text{p-Ps}} = 7989.42\,\mu\text{s}^{-1} + \Gamma^{(0)}_p B_p \left(\frac{\alpha}{\pi}\right)^2, \tag{4}$$

$$\Gamma^{\text{theory}}_{\text{o-Ps}} = 7.0382\,\mu\text{s}^{-1} + \Gamma^{(0)}_o B_o \left(\frac{\alpha}{\pi}\right)^2. \tag{5}$$

How do these predictions compare with experiments? For p-Ps the most recent result [10]

$$\Gamma^{\text{exp}}_{\text{p-Ps}} = 7990.9(1.7)\,\mu\text{s}^{-1} \tag{6}$$

is in good agreement with (4) if B_p is not too large.

For orthopositronium the situation is not clear. Precise experiments of the Ann Arbor group [11,12] found

$$\begin{aligned}\Gamma^{\text{exp}}_{\text{o-Ps}}(\text{gas measurement}) &= 7.0514(14)\,\mu\text{s}^{-1},\\ \Gamma^{\text{exp}}_{\text{o-Ps}}(\text{vacuum measurement}) &= 7.0482(16)\,\mu\text{s}^{-1},\end{aligned} \tag{7}$$

which, for $B_o = 0$, differ from (5) by 9.4σ and 6.3σ respectively. This apparent disagreement of experiment with theory has been known as the "orthopositronium lifetime puzzle." It should, however, be noted, that a more recent Tokyo result [13],

$$\Gamma^{\text{exp}}_{\text{o-Ps}}(\text{SiO}_2\text{ measurement}) = 7.0398(29)\,\mu\text{s}^{-1}, \tag{8}$$

agrees with the theory prediction if B_o is not too large. Both Tokyo and Ann Arbor groups are working to improve their results.

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Should future experimental efforts confirm the Ann Arbor results (7), in disagreement with the QED prediction (5), the orthopositronium lifetime puzzle could be solved if B_o turns out to be unusually large, e.g. ~ 250 for the vacuum measurement. Alternatively, one might speculate that some "New Physics" effects such as o-Ps decays involving axions, millicharged particles, etc., cause the excess of the measured decay rate over the QED predictions. Some of those exotic scenarios seem to have already been excluded by dedicated experimental studies. (For a review and references to original papers see e.g. [14].)

It was anticipated [10] that a full two-loop calculation might first be done for p-Ps. In fact the relative theoretical simplicity of p-Ps motivated the most recent lifetime measurement [10]. In this paper we present a complete calculation of B_p. Our result permits the rigorous test of bound state QED envisioned in [10]. We find that the two-loop non-logarithmic term in (1) has a small coefficient

$$B_p = 1.73(30), \tag{9}$$

and the theoretical prediction for the p-Ps lifetime becomes

$$\Gamma^{\text{theory}}_{\text{p-Ps}} = 7989.50(2)\,\mu\text{s}^{-1}. \tag{10}$$

Below we briefly discuss some details of our calculation.

The decay width of pPs $\to 2\gamma$ can be written as

$$\Gamma = \frac{1}{2!\,4\pi^2}\sum_{\{\lambda\}}\int \frac{\mathrm{d}^3k_1}{2\omega_1}\frac{\mathrm{d}^3k_2}{2\omega_2}\delta^4(P-k_1-k_2) \times\left|\int\frac{d^3p}{(2\pi)^3}\mathrm{Tr}\left[A(\lambda,\boldsymbol{p})\frac{1+\gamma_0}{2\sqrt{2}}\gamma_5\right]\ \phi(\boldsymbol{p})\right|^2, \tag{11}$$

where P is the four-momentum of the p-Ps, $\phi(\boldsymbol{p})$ is its wave function, and $A(\lambda,\boldsymbol{p})$ is the annihilation amplitude of an e^+e^- pair into a pair of photons with polarization $\{\lambda\}$.

In the non-covariant perturbation theory the on-shell amplitude of the process $e^+e^- \to \gamma\gamma$ reads

$$A = \frac{8\pi\alpha E_p}{E^2_{p-k}(E_p+m)}v^\dagger\Lambda_-(\boldsymbol{p})\vec{\alpha}\cdot\boldsymbol{e}_2\left[\vec{\alpha}\cdot(\boldsymbol{p}-\boldsymbol{k})+\beta m\right] \times\vec{\alpha}\cdot\boldsymbol{e}_1\Lambda_+(\boldsymbol{p})w + (\boldsymbol{e}_2\leftrightarrow\boldsymbol{e}_1, \boldsymbol{k}\leftrightarrow-\boldsymbol{k}). \tag{12}$$

Here $E_p = \sqrt{m^2+\boldsymbol{p}^2}$; $\boldsymbol{p}$ and $\boldsymbol{k}$ are electron and photon three-momenta in the p-Ps rest frame; w and v are bispinors of, respectively, electron and positron at rest, and $\Lambda_\pm(\boldsymbol{p})$ are the projectors on the positive and negative energy states,

$$\Lambda_\pm = \frac{1}{2}\left(1\pm\frac{\vec{\alpha}\boldsymbol{p}+\beta m}{E_p}\right). \tag{13}$$

To leading order one can neglect the small momenta $p\sim m\alpha$ compared to m and $|\boldsymbol{k}|\sim m$. One finds the following leading order amplitude

$$A_{\text{LO}} = -\frac{4\pi\alpha}{m^2}v^+(\vec{\alpha}\boldsymbol{e}_2)(\vec{\alpha}\boldsymbol{k})(\vec{\alpha}\boldsymbol{e}_1)w, \tag{14}$$

and the lowest order decay width

$$\Gamma_{\text{LO}} = \frac{4\pi\alpha^2}{m^2}\left|\psi(0)\right|^2 = \frac{m\alpha^5}{2}. \tag{15}$$

Higher order corrections to Γ_{LO} will be calculated using non-relativistic QED (NRQED) [15] with dimensional regularization [16]. We divide up the corrections into three parts:

$$B_p = B_p^{\text{squared}} + B_p^{\text{hard}} + B_p^{\text{soft}}, \tag{16}$$

where B_p^{squared} is the contribution of the one-loop amplitude squared and $B_p^{\text{hard, soft}}$ are the hard and soft contributions. The hard corrections arise as contributions of virtual photon momenta $k\sim m$. Their effects are described by adding operators containing $\delta(\boldsymbol{r})$ to the non-relativistic Hamiltonian. The technical challenge is to compute the Wilson coefficients of those operators. Fortunately those coefficients can be obtained using any convenient external states. In particular one can compute them for the electron and positron at rest. It is important to employ dimensional regularization, so that one avoids the necessity of subtracting the Coulomb singularities from box graphs. On the other hand, the soft contributions come from the region of virtual photon momenta of the order of $k\sim m\alpha$ and are sensitive to bound state dynamics. The actual calculation is briefly described below.

The square of the one-loop amplitude is easily obtained from the one-loop result:

$$B_p^{\text{squared}} = \left(\frac{5}{2}-\frac{\pi^2}{8}\right)^2 \simeq 1.6035. \tag{17}$$

B_p^{hard} consists of three types of contributions: vacuum polarization insertions in the photon propagators, light-by-light scattering diagrams, and two-photon corrections to the annihilation amplitude,

$$B_p^{\text{hard}} = B_p^{\text{hard}}(\text{VP}) + B_p^{\text{hard}}(\text{LL}) + B_p^{\text{hard}}(\gamma\gamma). \tag{18}$$

Vacuum polarization insertions into the one-loop graphs (an example is shown in Fig. 1(a)) were computed in [17,18],

$$B_p^{\text{hard}}(\text{VP}) = 0.4468(3). \tag{19}$$

Light-by-light scattering contributions (for examples see Fig. 1(b,c)) are more difficult to compute because of their imaginary parts which make numerical integration unstable. We computed them analytically, by formally assigning a large mass M to the internal fermions and expanding in $x = m/M$. Several terms in such an expansion are sufficient to find the result at $x=1$:

$$B_p^{\text{hard}}(\text{LL}) = -2.10(12). \tag{20}$$

The most difficult class of effects comes from the two-photon corrections, examples of which are shown in Fig. 1(d-f). We proceed in the following way: combine propagators using Feynman parameters; perform momentum integrations analytically; extract ultraviolet (UV) and infrared (IR) divergent pieces; integrate numerically over typically 5 (in some cases 6) Feynman parameters in the finite expressions.

Extraction of UV divergences is relatively simple – they show up, roughly speaking, as singularities in the overall factors rather than as divergent integrals over Feynman parameters.

Calculation of IR divergent diagrams is more demanding. For each such diagram we add and subtract a simpler diagram so that the sum is IR finite, and the subtraction can be calculated analytically. As an example let us consider the diagram shown in Fig. 1(d). The IR singularity in this diagram appears when momenta of the virtual photons are small. To suppress the contribution of this region we rewrite the propagator of the t-channel electron by subtracting its value and a suitable number of derivatives when both virtual photon momenta are zero. This difference leads to an infrared finite expression which can be computed numerically. To compensate this subtraction we must add the same diagram with the t-channel propagator replaced by a constant, $1/(2m^2)$. Such diagrams were studied in [19] and can be computed analytically.

Similar tricks are used to compute all other IR divergent diagrams, although the subtraction procedure is more tedious in case of diagrams with stronger singularities, like the planar box in Fig. 1(e).

Adding all two-photon diagrams and summing up numerical errors of individual diagrams in quadrature we find

$$B_p^{\rm hard}(\gamma\gamma) = -\frac{\pi^2}{2\epsilon} + 2\pi^2 \ln m - 42.23(27), \qquad (21)$$

and adding eqs. (19), (20), and (21), we find the total hard corrections to be

$$B_p^{\rm hard} = -\frac{\pi^2}{2\epsilon} + 2\pi^2 \ln m - 43.88(30). \qquad (22)$$

To calculate the soft scale contributions, one should account for the relativistic corrections to the annihilation amplitude (AA) $e^+e^- \to \gamma\gamma$ and to the positronium wave function (WF):

$$B_p^{\rm soft} = B_p^{\rm soft}(AA) + B_p^{\rm soft}(WF). \qquad (23)$$

For the annihilation amplitude correction, one expands the on-shell amplitude (12) to relative order $\mathcal{O}(p^2/m^2)$. Although the resulting integral is linearly divergent, using dimensional regularization one finds a finite result (see [20] for a discussion of this effect):

$$B_p^{\rm soft}(AA) = \frac{\pi^2}{3}. \qquad (24)$$

Relativistic corrections to the positronium wave function can be computed using the Breit Hamiltonian. Since we regularize all divergences dimensionally, we need the Breit Hamiltonian in d-dimensions derived in [20]. Its projection on the S-states can be found in eq. (39) of that paper. Performing calculations similar to those described after that equation, we find the wave function correction to the decay rate:

$$B_p^{\rm soft}(WF) + 2\pi^2 \ln\frac{1}{\alpha} = \frac{\pi^2}{2\epsilon} + 2\pi^2 \ln\frac{1}{m\alpha} + \frac{33\pi^2}{8}, \qquad (25)$$

where on the LHS we have separated the logarithm, to be consistent with division of corrections introduced in eq. (1).

The sum of the corrections to the annihilation amplitude (24) and to the wave function (25) gives the final result for the soft contributions,

$$B_p^{\rm soft} = \frac{\pi^2}{2\epsilon} - 2\pi^2 \ln m + \frac{107\pi^2}{24}. \qquad (26)$$

We note that this partial result cannot be directly compared to the soft corrections found in a previous study [7] since a different regularization scheme was employed there.

The final result in eq. (9), $B_p = 1.73(30)$, is obtained as a sum of the square of the one-loop corrections (17), and the genuine two-loop hard (22) and soft (26) contributions. We can now present a theoretical prediction for the 2-photon width of parapositronium with the 2-loop accuracy:

$$\begin{aligned}\Gamma^{\rm theory}_{\rm p-Ps} &= \frac{m\alpha^5}{2}\left[1 - \left(5 - \frac{\pi^2}{4}\right)\frac{\alpha}{\pi} + 2\alpha^2 \ln\frac{1}{\alpha}\right.\\ &\qquad \left. + 1.73(30)\left(\frac{\alpha}{\pi}\right)^2 - \frac{3\alpha^3}{2\pi}\ln^2\frac{1}{\alpha}\right]\\ &= 7989.50(2)\,\mu{\rm s}^{-1}. \end{aligned} \qquad (27)$$

Our final result (27) agrees well within $1\,\sigma$ with the most recent experimental result, eq. (6).

p-Ps can also decay into 4 or more photons. Those effects increase the p-Ps width by about $0.01\,\mu{\rm s}^{-1}$ (see [21] and references therein).

The coefficient of the non-logarithmic $(\alpha/\pi)^2$ term in eq. (27) is rather small, due to an almost complete cancelation between the soft and hard corrections. As we have already mentioned, only the sum of the two is regularization scheme independent and hence unambiguous. For this reason, it is likely that the cancellation between soft and hard pieces is not accidental. Scheme and gauge independent corrections, for example eqs. (17, 19, 20), seem to indicate that "natural scale" of the $\mathcal{O}(\alpha^2)$ corrections is [several units] $\times$ $(\alpha/\pi)^2$. We note that the result of our calculation, $B_p = 1.73(30)$, is much smaller than the estimate 40 ± 20 given in [7].

Having for the first time a complete two-loop correction to a QED bound state decay, one is tempted to speculate about the size of such corrections to the orthopositronium lifetime. Although nothing can be said rigorously, one could argue that most known second order effects have similar order of magnitude for o-Ps and p-Ps. A possible enhancement may be due to the larger (by about a factor of 3) number of Feynman diagrams contributing to o-Ps decay compared to p-Ps decay (this is already seen in the magnitudes of the $\mathcal{O}(\alpha)$ corrections). Unless this enhancement is dramatic for the complete $\mathcal{O}(\alpha^2)$ corrections to the decay rates, the theoretical prediction for the o-Ps lifetime will remain distant from the experimental results in eq. (7). It is therefore extremely important that the 3-photon decay of o-Ps be further studied experimentally.

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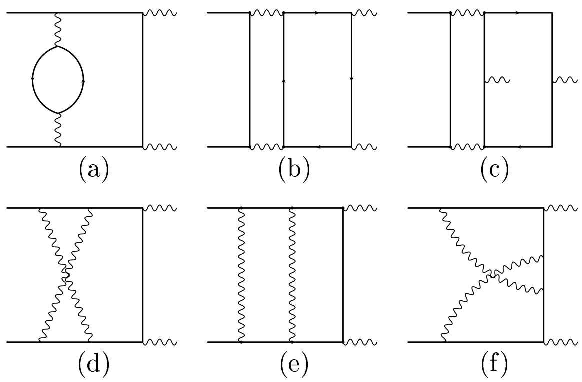

FIG. 1. Examples of two-loop hard corrections to p-Ps decay into two photons.