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$\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections to positronium energy levels

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Abstract

We present a calculation of the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections to positronium energy levels. The result is used to estimate the current uncertainty in theoretical predictions of the positronium spectrum.
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1. Recently, a calculation of the positronium spectrum with $\mathcal{O}(m\alpha^6)$ accuracy was completed [1,2]. The remaining uncertainty in theoretical predictions was estimated in [2] using the value of the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ leading logarithmic corrections to positronium energy levels:

$$\delta E = -\left(\frac{193}{5} + 7\boldsymbol{\sigma}\boldsymbol{\sigma}'\right) \frac{m\alpha^7 \ln^2 \alpha}{32\pi n^3} \delta_{l0}. \tag{1}$$

The purpose of this Letter is to present a detailed derivation of this result.

In general, an appearance of logarithms of the fine structure constant in QED bound state problems is related to the fact that several momentum scales are involved in bound state calculations. Contributions that are logarithmic in α, usually appear as integrals of the form:

$$\int \frac{d^3k}{(2\pi)^3} F(k), \tag{2}$$

with $F(k) \sim k^{-3}$ for the values of k such that $m\alpha \ll k \ll m$ or $m\alpha^2 \ll k \ll m\alpha$ [3]. Given the inequality $k \ll m$, it is possible to determine the leading logarithmic corrections

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in the framework of the nonrelativistic Quantum Mechanics by expanding all perturbations in series of k/m and by using the time-independent perturbation theory.

In order to find $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections, we first calculate all effective operators which deliver $\mathcal{O}(m\alpha^n \ln \alpha)$ contributions for $n < 7$. Such contributions first appear for $n = 5$ and $n = 6$. Then, the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections to energy levels are found by calculating relativistic corrections to these lower order operators. Some new operators arising at $\mathcal{O}(m\alpha^7)$ order should be also taken into account. For completeness we consider here the bound state of two different particles with masses m and M.

2. Let us first consider the self energy operator of one particle in the Coulomb field of the other. The scattering amplitude[1] reads:

$$-A_{\rm SE} = -\alpha \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{2k} j_i(\boldsymbol{p}', \boldsymbol{p}' - \boldsymbol{k}) e^{-i\boldsymbol{k}\boldsymbol{r}'_e} \frac{\delta_{ij} - \frac{k_i k_j}{k^2}}{k + H - E} j_j(\boldsymbol{p} - \boldsymbol{k}, \boldsymbol{p}) e^{i\boldsymbol{k}\boldsymbol{r}_e}. \tag{3}$$

The leading order logarithmic contribution is obtained by expanding the denominator $(k + H - E)^{-1}$ in $(H - E)/k$ and by using the leading non-relativistic approximation for the currents j_i. Only spin-independent part of the currents should be considered since the spin-dependent part contains additional powers of k which destroy the logarithmic integration. One obtains:

$$V_{\rm SE}^{\rm LO} \to \int \frac{d^3k}{(2\pi)^3} \frac{2\pi\alpha}{k^3} \frac{\boldsymbol{p}_k e^{-i\boldsymbol{k}\boldsymbol{r}_e}}{m} (H - E) \frac{\boldsymbol{p}_k e^{i\boldsymbol{k}\boldsymbol{r}_e}}{m}, \qquad \boldsymbol{p}_k \equiv \boldsymbol{p} - \boldsymbol{k}(\boldsymbol{p}\boldsymbol{k})/k^2. \tag{4}$$

It is easy to see that the integration over k in the above equation is already logarithmic. For this reason we can neglect the non-commutativity of the Hamiltonian H with the exponent $\exp(i\boldsymbol{k}\boldsymbol{r}_e)$. Fusing two exponents together, we integrate over k cutting the integral by the reduced particle mass $\mu = mM/(M + m)$ from above and by the particle energy $\sim p^2/\mu$ from below. Including the self energy of the second particle, we obtain an effective operator:

$$V_{\rm SE}^{\rm LO}(\boldsymbol{p}, \boldsymbol{r}) \to -\frac{2\alpha}{3\pi\mu^2} \left(1 - 2\frac{\mu^2}{mM}\right) \boldsymbol{p}(H - E)\boldsymbol{p} \ln \frac{p^2}{\mu^2}, \tag{5}$$

Here and below we consider $\ln(p^2/\mu^2)$ as commuting with all other quantities.

By averaging this operator over nS states, one obtains the well-known non-recoil logarithmic contribution to the Lamb shift of the nS levels:

$$\delta_{\rm SE}^{\rm LO} E = -\frac{8\alpha^2 \psi(0)^2}{3\mu^2} \left(1 - 2\frac{\mu^2}{mM}\right) \ln \alpha. \tag{6}$$

In order to obtain $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections induced by the operator $V_{\rm SE}^{\rm LO}$ it is necessary to calculate relativistic corrections to the expectation value of this operator. There are several sources of such corrections; below we analyze them.

The simplest one is the relativistic correction to the currents. It is obtained by the substitution

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[1] The notations of [2] are used throughout this paper.

$$\boldsymbol{j}(\boldsymbol{p}',\boldsymbol{p}) \approx \frac{\boldsymbol{p}'+\boldsymbol{p}}{2m} \to -\frac{p'^2+p^2}{4m^2}\frac{\boldsymbol{p}'+\boldsymbol{p}}{2m} \tag{7}$$

performed for one of the currents in Eq.(4). Again, only the orbital part of the currents contributes. The correction to the energy levels reads:

$$\begin{aligned}
\delta^{\rm curr}_{\rm SE}E &= \left\langle \int \frac{d^3k}{(2\pi)^3}\frac{4\pi\alpha}{2k^3}\left[H, \frac{\boldsymbol{p}_k e^{-i\boldsymbol{k}\boldsymbol{r}_e}}{m}\right]\frac{\boldsymbol{p}_k e^{i\boldsymbol{k}\boldsymbol{r}_e}}{m}\frac{p^2}{m^2}\right\rangle \\
&\to -\left\langle \int \frac{d^3k}{(2\pi)^3}\frac{8\pi\alpha\mu}{3m^4k^3}C(r)[\boldsymbol{p}, C(r)]\boldsymbol{p}\right\rangle \\
&\to \frac{16\alpha^4\psi(0)^2}{3\mu^2}\left(1 - 4\frac{\mu^2}{mM} + 2\frac{\mu^4}{m^2M^2}\right)\ln^2\alpha.
\end{aligned} \tag{8}$$

At the final stage of the calculation the self energy of the second particle was included.

The next-to-next-to-leading order effect of the retardation is described by the operator

$$V^{\rm ret}_{\rm SE} = -\int \frac{d^3k}{(2\pi)^3}\frac{\pi\alpha}{k^5}\left[H, \frac{\boldsymbol{p}_k e^{-i\boldsymbol{k}\boldsymbol{r}_e}}{m}\right]\left[H, \left[H, \frac{\boldsymbol{p}_k e^{i\boldsymbol{k}\boldsymbol{r}_e}}{m}\right]\right] + \text{h.c.} \tag{9}$$

Though there is the fifth power of k in the denominator, the spin-dependent part of the currents still does not contribute to the double logarithmic correction. It can be easily seen, that the commutator $[H, \boldsymbol{\sigma}\times\boldsymbol{k}\exp(i\boldsymbol{k}\boldsymbol{r}_e)]$ is $\mathcal{O}(k^2)$ as $k\to 0$, so that the resulting integral over k for the spin-dependent part of the currents is non-logarithmic. We therefore calculate the commutators in Eq. (9) and keep there only such terms that are quadratic in k. We obtain:

$$\begin{aligned}
\delta^{\rm ret}_{\rm SE}E &= \left\langle \int \frac{d^3k}{(2\pi)^3}\frac{4\pi\alpha}{3m^3k^3}\left(3+\frac{\mu}{m}\right)C(r)[\boldsymbol{p}, C(r)]\boldsymbol{p}\right\rangle + (m \leftrightarrow M) \\
&\to -\frac{8\alpha^4\psi(0)^2}{3\mu^2}\left(4 - 13\frac{\mu^2}{mM} + 2\frac{\mu^4}{m^2M^2}\right)\ln^2\alpha.
\end{aligned} \tag{10}$$

Next, we consider relativistic corrections to the Coulomb interaction and to the dispersion law of the particles in the intermediate state:

$$\int \frac{d^3k}{(2\pi)^3}\frac{2\pi\alpha}{k^3}\frac{\boldsymbol{p}_k e^{-i\boldsymbol{k}\boldsymbol{r}_e}}{m}\left[\frac{\boldsymbol{p}^4}{8m^3} + \frac{\boldsymbol{p}^4}{8M^3} + \frac{\pi\alpha}{2}\left(\frac{1}{m^2}+\frac{1}{M^2}\right)\delta(\boldsymbol{r})\right]\frac{\boldsymbol{p}_k e^{i\boldsymbol{k}\boldsymbol{r}_e}}{m}. \tag{11}$$

Taking the average value of this operator, one sees that the double logarithmic contribution is absent.

Accounting for additional magnetic exchange between the particles requires some care. There exist eight irreducible diagrams. Only four of them, shown in Fig.1, deliver double logarithmic contributions. One finds, that the double logarithmic contributions from the diagrams Fig.1a and Fig.1b compensate each other, while that of Fig.1c and Fig.1d sum up to the following energy shift:

$$\begin{aligned}
\delta_{\rm SE}^{\rm magn}E &= -\frac{\alpha}{2mM}\left\langle\int\frac{d^3k}{(2\pi)^3}\frac{4\pi\alpha}{3k^3}\frac{\boldsymbol{p}}{m}\frac{1}{r}p^2\frac{\boldsymbol{p}}{m}\right\rangle + (m\leftrightarrow M)\\
&\to -\frac{4\mu^2\alpha^4}{3\pi mM}\left(1-2\frac{\mu^2}{mM}\right)\left\langle\frac{\ln(\mu r)}{r^3}\right\rangle\\
&\to -\frac{8\alpha^4\psi(0)^2}{3mM}\left(1-2\frac{\mu^2}{mM}\right)\ln^2\alpha. \qquad (12)
\end{aligned}$$

Integration over k is performed in the limits $(\mu r^2)^{-1} < k < \mu$.

Finally, there exists $\mathcal{O}(\alpha^2)$ correction to the wave function of the bound state, i.e. the correction due to the iteration of the Breit Hamiltonian and the LO operator $V_{\rm SE}^{\rm LO}(\boldsymbol{p},\boldsymbol{r})$ (cf. Eq.(5)). The calculation of this correction is described in the final part of this Letter, where all lowest order logarithmic operators are considered simultaneously.

3. We consider now an exchange of a single magnetic photon between the two particles. The corresponding scattering amplitude is similar to Eq. (3):

$$-A_{\rm M} = \alpha\int\frac{d^3q}{(2\pi)^3}\frac{4\pi}{2q}J_i(-\boldsymbol{p}',-\boldsymbol{p}'-\boldsymbol{q})e^{-i\boldsymbol{q}\boldsymbol{r}'_p}\frac{\delta_{ij}-\frac{q_iq_j}{q^2}}{q+H-E}j_j(\boldsymbol{p}-\boldsymbol{q},\boldsymbol{p})e^{i\boldsymbol{q}\boldsymbol{r}_e}+\text{h.c.} \qquad (13)$$

We used this amplitude in [2] when discussed the retardation effects. Considering the next-to-leading order retardation, one finds the operator

$$V_{\rm M}^{\rm LO} \to -\int\frac{d^3q}{(2\pi)^3}\frac{4\pi\alpha}{mMq^3}e^{i\boldsymbol{q}\boldsymbol{r}}\boldsymbol{p}_q(H-E)\boldsymbol{p}_q, \qquad (14)$$

where again the non-commutativity of $e^{i\boldsymbol{q}\boldsymbol{r}}$ and H has been neglected. Because of the presence of the $\exp(i\boldsymbol{q}\boldsymbol{r})$, the logarithmic integral over q in Eq.(14) is cut at $1/r\sim p$ from above [3]. We arrive at the following effective operator:

$$V_{\rm M}^{\rm LO}(\boldsymbol{p},\boldsymbol{r}) = -\frac{2\alpha}{3\pi mM}\boldsymbol{p}(H-E)\boldsymbol{p}\ln\frac{p^2}{\mu^2}. \qquad (15)$$

The sources of double logarithmic corrections to the single magnetic exchange are the same as in the case of the self energy operator. The only essential difference in two calculations consists in a change of the upper cut-off. For this reason we skip a detailed discussion and present the results of the calculation. We obtain:

$$\delta_{\rm M}^{\rm curr}E = \frac{8\alpha^4\psi(0)^2}{3mM}\left(1-2\frac{\mu^2}{mM}\right)\ln^2\alpha, \qquad (16)$$

$$\delta_{\rm M}^{\rm ret}E = -\frac{16\alpha^4\psi(0)^2}{15mM}\left(1-\frac{5}{2}\frac{\mu^2}{mM}\right)\ln^2\alpha, \qquad (17)$$

$$\delta_{\rm M}^{\rm magn}E = -\frac{8\alpha^4\psi(0)^2\mu^2}{3m^2M^2}\ln^2\alpha. \qquad (18)$$

4. The next source of the double logarithmic corrections is the double magnetic exchange with two seagull vertices. The corresponding potential reads

$$V_{\rm SS} = -\frac{\alpha^2}{mM}\int\frac{d^3k}{(2\pi)^3}\frac{4\pi}{2k}\frac{4\pi}{2k'}e^{i\boldsymbol{q}\boldsymbol{r}_p}\frac{(\boldsymbol{e}\boldsymbol{e}')^2}{k+k'+H-E}e^{-i\boldsymbol{q}\boldsymbol{r}_e}. \tag{19}$$

Here $\boldsymbol{k}' = \boldsymbol{q} - \boldsymbol{k}$, $e_i e_j = \delta_{ij} - k_i k_j/k^2$ and similarly for $\boldsymbol{e}'$. It is sufficient to consider the leading nonrelativistic approximation for the seagull vertex. From Eq.(19) one sees, that the logarithmic contribution comes from the region of momenta where $|\boldsymbol{q}| \ll |\boldsymbol{k}| \ll \mu$. Neglecting $H - E$ in comparison with k and integrating over k from q to μ one obtains:

$$V_{\rm SS}^{\rm LO} = \frac{2\alpha^2}{mM}\ln\frac{q}{\mu}. \tag{20}$$

We consider now relativistic corrections to the operator in Eq.(20). In this case there are other sources of corrections, as compared to two cases considered above. The following corrections should be considered – relativistic corrections to the seagull vertex, an appearance of the magnetic-magnetic-Coulomb vertex, an expansion of the "heavy" intermediate energy denominators in powers of $|\boldsymbol{p}|/m, k/m, k'/m$ and the usual[2] retardation effects. We find that none of these effects produces the double logarithmic correction.

For example, relativistic corrections to the seagull vertex are polynomial in the momenta p, k and q. On the other hand, the structure of the denominator in Eq.(19) does not change. In this case the logarithmic contribution comes only from the region of large $|\boldsymbol{k}|$, so that the resulting operator is of the form of Eq.(20) times a polynomial in external momenta. Such operators are too singular to produce a double logarithmic contribution.

Hence we conclude that the $\mathcal{O}(m\alpha^7\ln^2\alpha)$ corrections to the expectation value of the operator Eq.(20) appear only due to relativistic corrections to the wave functions. As we have mentioned already, these corrections are considered at the end of this Letter.

5. We now turn to logarithmic operators which arise at the $\mathcal{O}(m\alpha^6)$ order. The $\mathcal{O}(\alpha)$ correction to such operators can be caused only by the effect of retardation. There are two effects of this sort. Let us denote by k the momentum of the photon in the intermediate state. Then, for the "light" $e^+e^-\gamma$ intermediate state the energy denominators are expanded in $(H-E)/k$, whereas for the "heavy" $e^+e^-e^+e^-\gamma$ intermediate state the expansion parameter is k/m. In contrast to these, all other relativistic corrections are of relative α^2 order. Retardation corrections clearly require that the corresponding effective operators are induced by an exchange of at least one magnetic photon. Inspecting such operators in [2], we observe that only two of them, the "one-loop" operator and the operator induced by a double magnetic exchange with one seagull vertex, can give rise to $\mathcal{O}(m\alpha^7\ln^2\alpha)$ corrections on one hand, and were not considered yet on the other.

Retardation correction to the one-loop operator cannot produce the double logarithmic contribution. Indeed, both effective vertices entering the one-loop operator, are $\boldsymbol{p}$-independent, so that their commutators with H produce only positive powers of momenta. One can easily check that resulting operators are too singular to produce two logarithms.

For the same reason considering the double magnetic exchange with one seagull vertex we keep orbital currents only. The sum of three diagrams (see Fig.2) then gives:

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[2]By "usual" we mean here such retardation effects, that do not resolve the point-like seagull vertex.

$$V_{\rm S}^{\rm 2ex} = \frac{\alpha^2}{m}\int \frac{d^3k}{(2\pi)^3}\frac{4\pi}{2k'}\frac{4\pi}{2k}\left\{v_2\frac{1}{k'+k+H-E}v_1\frac{1}{k+H-E}v_3\right.$$
$$\left. +v_1\frac{1}{k'+H-E}v_2\frac{1}{k+H-E}v_3 + v_1\frac{1}{k'+H-E}v_3\frac{1}{k'+k+H-E}v_2\right\}, \qquad (21)$$

where

$$v_1 = \frac{\boldsymbol{p}'\boldsymbol{e}'}{M}e^{i\boldsymbol{k}'\boldsymbol{r}'_p}, \qquad v_2 = \boldsymbol{e}'\boldsymbol{e}e^{i\boldsymbol{q}\boldsymbol{r}_e}, \qquad v_3 = \frac{\boldsymbol{p}\boldsymbol{e}}{M}e^{i\boldsymbol{k}\boldsymbol{r}_p}. \qquad (22)$$

The contribution of three diagrams with the seagull vertex on the second particle line is obtained from Eq.(21) by the substitution $m \leftrightarrow M$. An account of the retardation to first order gives the operator:

$$V_{\rm S}^{\rm 2ex} \to -\frac{\alpha^2}{m}\int \frac{d^3k}{(2\pi)^3}\frac{4\pi}{2k'}\frac{4\pi}{2k}\left\{v_2\frac{1}{k'+k}v_1\frac{1}{k^2}[H,v_3] + v_1\frac{1}{k'}v_2\frac{1}{k^2}[H,v_3]\right.$$
$$\left. +v_1\frac{1}{k'}v_3\frac{1}{(k'+k)^2}[H,v_2]\right\} + \text{h.c.}. \qquad (23)$$

For $k \ll q$ this operator reduces to

$$V_{\rm S}^{\rm 2ex} = \frac{\alpha^3}{3mM^2}\int \frac{d^3k}{(2\pi)^3}\frac{4\pi}{k^3}\frac{4\pi}{q^2}\boldsymbol{p}_q[\boldsymbol{p}_q, C(r)] + \text{h.c.} + (m \leftrightarrow M). \qquad (24)$$

Double logarithmic correction to the energy arises if one integrates over k from q^2/μ to q, and then over q from $\mu\alpha$ to μ. The result is

$$\delta_{\rm S}^{\rm 2ex}E = -\frac{8\alpha^4\psi(0)^2}{3mM}\ln^2\alpha. \qquad (25)$$

The last "irreducible" correction arises due to the single-seagull diagrams with one of the magnetic quanta absorbed by the same charged particle (see Fig.3). The only difference with the previous calculation is the upper cut-off for the integral over k in the contribution of the diagram Fig.3a. Now this cut-off equals μ. We also note that the diagram Fig.3b does not deliver a double logarithmic contribution. The result is:

$$\delta_{\rm S}^{\rm 1ex}E = -\frac{4\alpha^4\psi(0)^2}{mM}\ln^2\alpha. \qquad (26)$$

Eq.(26) completes the analysis of the "irreducible" $\mathcal{O}(m\alpha^7\ln^2\alpha)$ corrections.

6. The last source of the $\mathcal{O}(m\alpha^7\ln^2\alpha)$ corrections is related to the wave function modification by relativistic effects:

$$\delta_\psi E = \langle VGU + UGV\rangle\,. \qquad (27)$$

Here G is the reduced nonrelativistic Green function, U is the Breit Hamiltonian projected on S-states:

$$U(\boldsymbol{p},\boldsymbol{r}) = -\frac{1-3\frac{\mu^2}{mM}}{2\mu}\left(\frac{p^2}{2\mu}\right)^2 + \frac{\mu}{mM}\left\{\frac{p^2}{2\mu}, C(r)\right\} + \frac{\pi\alpha}{2\mu^2}\left(1+2\frac{\mu^2}{mM}\left[1+\frac{2}{3}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right]\right)\delta(\boldsymbol{r}), \tag{28}$$

and the operator V is the sum of the lowest-order logarithmic operators Eqs.(5,15,20):

$$V(\boldsymbol{p},\boldsymbol{r}) = \frac{2\alpha}{3\pi\mu^2}\ln\frac{p^2}{\mu^2}\left\{\left(1-\frac{\mu^2}{mM}\right)\boldsymbol{p}(H-E)\boldsymbol{p} - \frac{3\pi\alpha}{2}\frac{\mu^2}{mM}\right\}. \tag{29}$$

The sum of the first two terms in $U(\boldsymbol{p},\boldsymbol{r})$ can be represented as a linear combination of the operators H^2, $\{H, C(r)\}$, and $C(r)^2$ (see [2]). The last two terms induce the following double logarithmic corrections:

$$\begin{aligned}
\delta_{\psi,1}E &= \frac{1-\frac{\mu^2}{mM}}{\mu}\langle\{H,C(r)\}GV\rangle \to -\frac{1-\frac{\mu^2}{mM}}{\mu}\langle C(r)V\rangle \\
&\to -\frac{2\alpha}{3\pi\mu^3}\left(1-\frac{\mu^2}{mM}\right)^2\left\langle \ln\frac{p^2}{\mu^2}C(r)\boldsymbol{p}[\boldsymbol{p},C(r)]\right\rangle \\
&\to \frac{16\alpha^4\psi(0)^2}{3\mu^2}\left(1-\frac{\mu^2}{mM}\right)^2\int\frac{d^3p}{(2\pi)^3}\frac{\pi^2}{p^3}\ln\frac{p^2}{\mu^2} \\
&\to -\frac{8\alpha^4\psi(0)^2}{3\mu^2}\left(1-\frac{\mu^2}{mM}\right)^2\ln^2\alpha;
\end{aligned} \tag{30}$$

$$\begin{aligned}
\delta_{\psi,2}E &= -\frac{1+\frac{\mu^2}{mM}}{\mu}\left\langle C(r)^2GV\right\rangle \\
&\to -\frac{1+\frac{\mu^2}{mM}}{\mu^3}\left\langle \ln\frac{p^2}{\mu^2}C(r)^2G\left\{\left(1-\frac{\mu^2}{mM}\right)\boldsymbol{p}[\boldsymbol{p},C(r)] + \frac{3\pi\alpha}{2}\frac{\mu^2}{mM}\right\}\right\rangle \\
&\to \frac{1+\frac{\mu^2}{mM}}{\mu^3}\left\langle \ln\frac{p^2}{\mu^2}C(r)^2G_0\left\{4\pi\alpha\left(1-\frac{\mu^2}{mM}\right) - \frac{3\pi\alpha}{2}\frac{\mu^2}{mM}\right\}\right\rangle \\
&\to \frac{16\alpha^4\psi(0)^2}{3\mu^2}\left(1-\frac{3}{8}\frac{\mu^2}{mM} - \frac{11}{8}\frac{\mu^4}{m^2M^2}\right)\ln^2\alpha.
\end{aligned} \tag{31}$$

In a similar manner we consider the contribution of the $\delta(\boldsymbol{r})$ operator from the Breit Hamiltonian. We obtain:

$$\begin{aligned}
\delta_{\psi,3}E &\to \frac{\pi\alpha}{\mu^2}\left(1+2\frac{\mu^2}{mM}\left[1+\frac{2}{3}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right]\right)\langle\delta(\boldsymbol{r})GV\rangle \\
&\to -\frac{4\alpha^4\psi(0)^2}{3\mu^2}\left(1+2\frac{\mu^2}{mM}\left[1+\frac{2}{3}\boldsymbol{\sigma}\boldsymbol{\sigma}'\right]\right)\left(1-\frac{7}{4}\frac{\mu^2}{mM}\right)\ln^2\alpha.
\end{aligned} \tag{32}$$

7. In order to check that we have accounted for all sources of the $\mathcal{O}(m\alpha^7\ln^2\alpha)$ corrections, it is useful to consider the problem from a more formal point of view. As we mentioned

above, logarithms of α arise because of the hierarchy of scales in QED bound states calculations. In the framework of NRQED such logarithms may be detected by analyzing the on-shell scattering amplitude of two particles and identifying such contributions to this amplitude that have zero index in the nonrelativistic region. Therefore, to get the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections to energy levels, we have to consider the $\mathcal{O}(\alpha^4)$ scattering amplitude and find all zero index graphs.

To do so, we proceed in the following way: i) divide all graphs into classes according to the number of magnetic photons (e.g., class 0 contains all graphs with four Coulomb exchanges, class 1 contains all graphs with three Coulomb and one magnetic photon, and so on); ii) in each class, we consider only such graphs that in the leading nonrelativistic approximation have negative or zero indices; iii) the negative indices are shifted to zero by accounting for retardation corrections as well as all possible relativistic corrections to the elements of a given graph.

Consider first the class 0. In the leading non-relativistic approximation the only contribution comes from the four-Coulomb ladder graph with both electron and positron staying in positive-energy intermediate states. This graph has the index $3 \times 3 - 3 \times 2 - 4 \times 2 = -5$, where the first term comes from three-loop integration volume, the second one from three energy denominators, corresponding to e^+e^- intermediate states and the last term arises from the four Coulomb propagators. To shift this index to zero, we have to consider relativistic corrections with the fifth net power of momenta. However, all relativistic effects in pure Coulomb graphs bring in corrections which scale like $\mathcal{O}(v^2)$, i.e. they increase the index by 2. Therefore, we conclude that the class 0 is free from potentially logarithmic graphs.

Turning to other classes, we notice that one can significantly decrease the number of graphs by considering only such where all magnetic and seagull vertices belong to one and the same fermion line. It is easy to see that the equal-time transfer of a vertex from one fermion line to the other does not change the index of the graph. Hence, for the purpose of index counting we can consider the simplified problem of the $\mathcal{O}(\alpha^4)$ scattering of one particle at the external Coulomb field. After identifying potentially logarithmic graphs with magnetic and seagull vertices on one fermion line one must consider all possible equal-time transfers of those vertices between two fermion lines and all possible relativistic corrections.

To make further discussion more transparent, we introduce the following notations. The Coulomb, magnetic, and seagull vertices are denoted by C, M, and S, respectively. For the class 1, in the leading nonrelativistic approximation we have the following graphs: MMCCC, CMMCC, CCMMC, and CCCMM[3]. Their index, -3, is the sum of 9 from the integration volume, 2 from two Pauli currents in magnetic vertices, –6 from three Coulomb propagators, –6 from three energy denominators related to intermediate states without photon, –1 from the $e^+e^-\gamma$ intermediate state, and –1 from the normalization factor of the propagator of the magnetic photon. To increase this index to zero it is necessary to account for the effects of retardation. Since each order of perturbation theory for the retardation increases the index by 1, we have to consider either the third order retardation effects or the first

[3]The notations are such, that e.g. MMCCC represents a graph were first the magnetic photon is emitted, then it is absorbed and then three Coulomb exchanges occur.

order retardation with additional $\mathcal{O}(v^2)$ relativistic corrections. Note that an account of the retardation includes not only an expansion of energy denominators $(k + \Delta E)^{-1}$, but also permutations of the Coulomb and magnetic vertices, e.g., MMCCC→MCMCC. By inspection we find that all these effects were considered in our previous analysis of single-magnetic contributions.

An analysis of the class 2 graphs is slightly more involved because of an appearance of the seagull vertex. The lowest index, –2, arises in the leading nonrelativistic approximation for the graphs with two seagull vertices, CCSS, CSSC, and SSCC. To compensate this index we have to account for the next-to-next-to-leading order retardation or some $\mathcal{O}(v^2)$ relativistic correction. The index –1 graphs include SMM, MSM, MMS, and $\mathrm{M}_i\mathrm{M}_i\mathrm{M}_j\mathrm{M}_j$ with two additional Coulomb vertices, none of which is located either between S and M or M_i and M_i. The next-to-leading order retardation shifts the index of these graphs to zero. Finally, there are index 0 graphs, $\mathrm{M}_i\mathrm{M}_j\mathrm{M}_j\mathrm{M}_i$ and $\mathrm{M}_j\mathrm{M}_i\mathrm{M}_j\mathrm{M}_i$ with two additional Coulomb photons, each located either to the left or to the right of all M-vertices. One can easily find the one-to-one correspondence between any of these graphs and some parts of the double-magnetic contributions considered in our previous analysis.

In the class 3 there are only index 0 graphs, SSMM and MMSS with one C outside pairs SS and MM. After equal-time transfer of one seagull and one magnetic vertex to the second fermion line, we get the graphic representation for the correction to the average value of the operator Eq.(20), induced by the magnetic part of the Breit Hamiltonian.

Finally, an inspection of the class 4 graphs shows that all of them have positive indices.

8. It remains to sum up all $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ contributions. We obtain:

$$\delta E_{\mathrm{aver}} = -\frac{\alpha^4 \psi(0)^2}{\mu^2} \left(4 - \frac{43}{5} \frac{\mu^2}{mM} \right) \ln^2 \alpha, \tag{33}$$

$$\delta E_{\mathrm{hfs}} = -\frac{64 \alpha^4 \psi(0)^2}{9mM} \left(1 - \frac{7}{4} \frac{\mu^2}{mM} \right) \ln^2 \alpha. \tag{34}$$

In the limiting case $m/M \to 0$, the old result for the hydrogen [5] is reproduced. Considering pure recoil corrections, one finds:

$$\delta E^{\mathrm{rec}}_{\mathrm{aver}} = -\frac{11 \alpha^4 \psi(0)^2}{15mM} \ln^2 \alpha, \qquad \delta E^{\mathrm{rec}}_{\mathrm{hfs}} = \frac{16 \alpha^4 \psi(0)^2 \mu^2}{3m^2 M^2} \ln^2 \alpha. \tag{35}$$

It is interesting to note, that the $\mathcal{O}(\mu^2/(mM)^2)$ terms mutually canceled not only in Eq.(33), but also in individual contributions, i.e. those induced by the self energy, the single magnetic exchange and the double magnetic exchange with one seagull vertex.

For the positronium, it is necessary to add the contribution caused by the virtual annihilation which is easily extracted from Eq. (32). One obtains:

$$\delta_{\mathrm{ann}} E = -3 \frac{3 + \boldsymbol{\sigma}\boldsymbol{\sigma}'}{4} \frac{\alpha^4 \psi(0)^2}{m^2} \ln^2 \alpha. \tag{36}$$

Using $\psi(0)^2 = \delta_{l0} m^3 \alpha^3 / (8\pi n^3)$, we arrive at the final result for the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ contribution to the positronium energy levels:

$$\delta E = -\left(\frac{193}{5} + 7 \boldsymbol{\sigma}\boldsymbol{\sigma}' \right) \frac{m\alpha^7 \ln^2 \alpha}{32\pi n^3} \delta_{l0}. \tag{37}$$

Its spin-dependent part is in agreement with the result of Ref. [4].

Numerically, the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ contribution to the triplet energy levels equals to

$$\delta E\left(n^3 S_1\right) = -\frac{1.5}{n^3}\ \mathrm{MHz}. \tag{38}$$

The correction to the difference $E(2^3S_1) - E(1^3S_1)$ amounts therefore to 1.3 MHz. For the singlet states, the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ correction gives:

$$\delta E\left(n^1 S_0\right) = -\frac{0.58}{n^3}\ \mathrm{MHz}. \tag{39}$$

We then find that the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ correction to the positronium ground state hyperfine splitting amounts to -0.9 MHz.

We conclude, that the $\mathcal{O}(m\alpha^7 \ln^2 \alpha)$ corrections turn out to be of the order of 1 MHz and hence somewhat enhanced, as compared to the naive estimate of the magnitude of the $m\alpha^7$ effects. We note, that at $O(m\alpha^6)$ the remaining, non-logarithmic corrections, contributed approximately one half of the leading logarithmic ones. Extrapolating this situation to order $\mathcal{O}(m\alpha^7)$, we conclude that the current theoretical uncertainty in the predictions for the positronium energy levels [1,2] should be approximately 0.5 MHz.

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REFERENCES

[1] K. Pachucki and S. G. Karshenboim, Phys. Rev. Lett. **80** (1998) 2101.
[2] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, hep-ph/9901394.
[3] I. B. Khriplovich, A. I. Milstein, and A. S. Yelkhovsky, Phys. Scr. **T46** (1993) 252.
[4] S. G. Karshenboim, JETP **76** (1993) 41.
[5] A. J. Layzer, Phys. Rev. Lett. **4** (1960) 580;
H. M. Fried and D. R. Yennie, Phys. Rev. Lett. **4** (1960) 583.

FIGURES

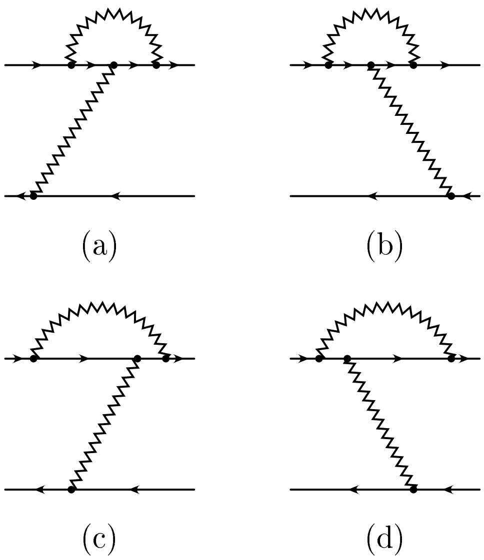

FIG. 1. Corrections to self-energy operator Eq.(5) due to additional magnetic exchange.

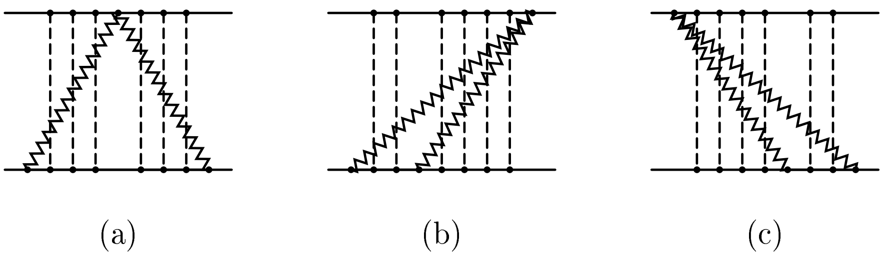

FIG. 2. Double magnetic exchange with one-seagull vertex. The dashed lines, representing Coulomb exchanges, show that the exact Green function for the system positronium plus photon(s) should be used in the calculation.

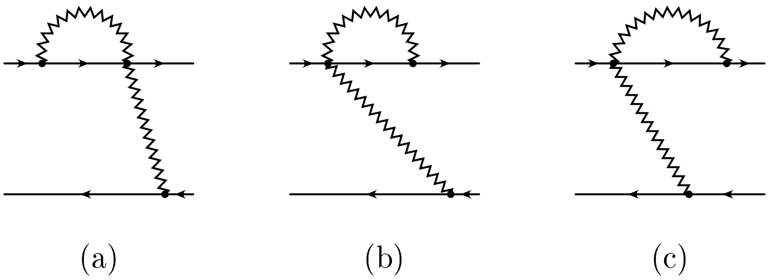

FIG. 3. Seagull corrections to single magnetic exchange.