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# Decoupling of heavy-quark loops in light-light and heavy-light quark currents

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**Abstract**

Matching of light-light quark currents in QCD with a heavy flavour and in the low-energy effective QCD is calculated in $\overline{\mathrm{MS}}$ at two loops. Heavy-light HQET currents are similarly considered.

Consider QCD with $n_l$ light flavours and a single heavy one having mass $M$. Processes involving only light fields with momenta $p_i \ll M$ can be described by an effective field theory — QCD with $n_l$ flavours plus $1/M^n$ suppressed local operators in the Lagrangian, which are the remnants of heavy quark loops shrinked to a point. This low-energy effective theory is constructed to reproduce $S$-matrix elements of the full theory expanded to some order in $p_i/M$. This condition fixes $\overline{\mathrm{MS}}$ coupling $\alpha'_s(\mu)$ of the effective theory, its gauge parameter $a'(\mu)$ and running light quark masses $m'_i(\mu)$ in terms of the parameters of the full theory. Two-loop matching was considered in [1, 2], and three-loop one — in [3].

Operators of the full QCD can be expanded in $1/M$ in operators of the effective theory. Coefficients of these expansions are fixed by equating on-shell matrix elements. In this Letter, I consider bilinear quark currents $j_n(\mu) = Z_n^{-1}(\mu) j_{n0}$, where the bare currents are $j_{n0} = \bar{q}_0 \gamma^{[\mu_1} \ldots \gamma^{\mu_n]} q_0$, and $Z_n(\mu)$ are their $\overline{\mathrm{MS}}$ renormalization constants. In the low-energy theory, they are equal to $C_n(\mu) j'_n(\mu)$ plus power-suppressed terms (which will be omitted throughout this paper). It is natural to match the currents at $\mu = M$, because $C_n(M)$ contains no large logarithms. Currents at arbitrary normalization scales can be related by

$$\exp\left(-\int_{\alpha_s(M)}^{\alpha_s(\mu)} \frac{\gamma_n(\alpha)}{2\beta(\alpha)} \frac{d\alpha}{\alpha}\right) j_n(\mu) = C_n(M) \exp\left(-\int_{\alpha'_s(M)}^{\alpha'_s(\mu')} \frac{\gamma'_n(\alpha)}{2\beta'(\alpha)} \frac{d\alpha}{\alpha}\right) j'_n(\mu'), \quad (1)$$

where

$$\gamma_n(\alpha_s) = \frac{d \log Z_n}{d \log \mu} = \gamma_{n0} \frac{\alpha_s}{4\pi} + \gamma_{n1} \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \quad (2)$$

is the current's anomalous dimension (obtained at two loops for generic $n$ in [4]), and

$$\frac{d \log \alpha_s}{d \log \mu} = -2\varepsilon - 2\beta(\alpha_s), \quad \beta(\alpha_s) = \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots, \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \quad (3)$$

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$T_F = \frac{1}{2}$, $C_A = N$, $C_F = (N^2-1)/(2N)$, $N = 3$ is the number of colours; $n_f = n_l + 1$; $d = 4 - 2\varepsilon$ is the space-time dimension; $\gamma'_n$, $\beta'$ are the corresponding quantities in $n_l$-flavour QCD.

On-shell matrix element of $j_n(\mu)$ is $M(\mu) = Z_q Z_n^{-1}(\mu)\Gamma_{n0}$, where $\Gamma_{n0}$ is the bare vertex function and $Z_q$ is the on-shell renormalization constant of the light-quark field. It should be equal to $C_n(\mu)M'_n(\mu)$ plus power-suppressed terms, where $M'_n(\mu) = Z'_q Z'^{-1}_n(\mu)\Gamma'_{n0}$. Both matrix elements are ultraviolet-finite; their infrared divergences coincide, because both theories are identical in the infrared region. Therefore (cf. [5, 4]),

$$C_n(\mu) = \frac{Z_q}{Z'_q}\frac{Z'_n(\mu)}{Z_n(\mu)}\frac{\Gamma_{n0}}{\Gamma'_{n0}}\,. \tag{4}$$

The on-shell wave-function renormalization constant is $Z_q = [1 - \Sigma_V(0)]^{-1}$, where the bare mass operator is $\Sigma(p) = \not{p}\Sigma_V(p^2)$. Only the heavy-quark loop contributes (Fig. 1$a$), and we obtain at two loops

$$\Sigma_V(0) = C_F T_F \frac{g_0^4 M^{-4\varepsilon}}{(4\pi)^d} I \frac{(d-1)(d-4)}{d}\,, \tag{5}$$

where

$$\begin{aligned}
&iC_F g_0^2 \int \frac{d^d k}{(2\pi)^d}\frac{\Pi(k^2)}{k^4} = C_F T_F \frac{g_0^4 M^{-4\varepsilon}}{(4\pi)^d} I\,,\\
&I = \Gamma^2(\varepsilon)\frac{2(d-6)}{(d-2)(d-5)(d-7)} = -\tfrac{2}{3}\Gamma^2(\varepsilon)\left(1 - \tfrac{2}{3}\varepsilon + \tfrac{22}{9}\varepsilon^2 + \cdots\right)\,,
\end{aligned} \tag{6}$$

and $\Pi(k^2)$ is the heavy-quark contribution to the gluon polarization operator. This agrees with the decoupling relation [3] $q = (Z_q/Z'_q)^{1/2} q'$.

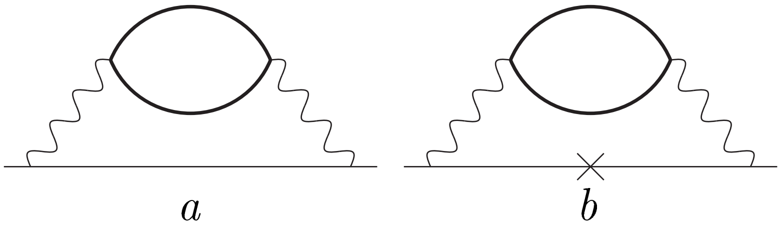

Figure 1: Mass operator and vertex function

The bare vertex function (Fig. 1$b$) can be calculated at any on-shell momenta of the quarks. It is most easy to set both momenta to zero (thus excludung power-suppressed terms in (4)):

$$\Gamma_{n0} = 1 - C_F T_F \frac{g_0^4 M^{-4\varepsilon}}{(4\pi)^d} I \left(\frac{(d-2n)^2}{d} - 1\right)\,. \tag{7}$$

The corresponding quantities in the low-energy theory are $Z'_q = 1$, $\Gamma'_{n0} = 1$. The ratio

$$\frac{Z_n}{Z'_n} = 1 + \frac{1}{4}\left(\gamma_{n0}\Delta\beta_0 - \Delta\gamma_{n1}\varepsilon\right)\left(\frac{\alpha_s}{4\pi\varepsilon}\right)^2 = 1 + \frac{1}{9}C_F T_F \left(\frac{\alpha_s}{4\pi\varepsilon}\right)^2 (n-1)\left[6(n-3) - (n-15)\varepsilon\right] \tag{8}$$

(where $\Delta\beta_0 = -\frac{4}{3}T_F$ and $\Delta\gamma_{n1}$ are the contributions of a single flavour) can be obtained from [4].

Using

$$\frac{g_0^2 M^{-2\varepsilon}\Gamma(1+\varepsilon)}{(4\pi)^{d/2}} = \frac{\alpha_s(M)}{4\pi} + \mathcal{O}(\alpha_s^2)\,, \tag{9}$$

we finally obtain

$$j_n(M) = \left[1 + C_F T_F \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{(n-1)(85n-267)}{54}\right] j_n'(M)\,. \tag{10}$$

To all orders, $j_1 = j_1'$ [5]; this is natural, because the integral of the vector current is the number of quarks minus antiquarks, which is the same in both theories. Also $m(\bar{q}q) = m'(\bar{q}q)'$ to all orders [6]. Therefore, $C_0$ can be obtained from the mass decoupling relation [1, 3]. The currents $j_3$ and $j_4$ differ from $j_1$ and $j_0$ by insertion of 't Hooft-Veltman $\gamma_5^{\mathrm{HV}}$; they differ from the corresponding currents with the anticommuting $\gamma_5^{\mathrm{AC}}$ by finite renormalizations $\bar{q}\gamma_5^{\mathrm{AC}}\gamma^\mu q = Z_A \bar{q}\gamma_5^{\mathrm{HV}}\gamma^\mu q$, $\bar{q}\gamma_5^{\mathrm{AC}} q = Z_P \bar{q}\gamma_5^{\mathrm{HV}} q$, known to three loops [7]:

$$\begin{aligned} Z_A &= 1 - 4C_F\frac{\alpha_s}{4\pi} + C_F\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{198C_F - 107C_A + 4T_F n_f}{9} + \cdots\,,\\ Z_P &= 1 - 8C_F\frac{\alpha_s}{4\pi} + C_F\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{2(C_A + 4T_F n_f)}{9} + \cdots \end{aligned} \tag{11}$$

Insertion of $\gamma_5^{\mathrm{AC}}$ does not change the matching coefficient. Hence, $C_3 = C_1 Z_A'/Z_A$, $C_4 = C_0 Z_P'/Z_P$, where $Z_A'$, $Z_P'$ contain $n_l$ instead of $n_f = n_l + 1$. The result for $n = 2$ is, to the best of my knowledge, new.

The HQET heavy-light current $\tilde{\jmath}(\mu) = \tilde{Z}^{-1}(\mu)\tilde{\jmath}_0$, $\tilde{\jmath}_0 = \bar{q}_0\tilde{Q}_0$ can be considered similarly (results don't depend on its $\gamma$-matrix structure). Instead of (4), we have now

$$\tilde{C}(\mu) = \left(\frac{Z_q}{Z_q'}\right)^{1/2} \left(\frac{\tilde{Z}_Q}{\tilde{Z}_Q'}\right)^{1/2} \frac{\tilde{Z}'(\mu)}{\tilde{Z}(\mu)} \frac{\tilde{\Gamma}_0}{\tilde{\Gamma}_0'}\,. \tag{12}$$

On-shell renormalization constant of the static quark field is $\tilde{Z}_Q = \left[1 - \left(d\tilde{\Sigma}/d\omega\right)_{\omega=0}\right]^{-1}$, where at two loops [4]

$$\left.\frac{d\tilde{\Sigma}}{d\omega}\right|_{\omega=0} = -C_F T_F \frac{g_0^4 M^{-4\varepsilon}}{(4\pi)^d}(d-1) I\,. \tag{13}$$

The bare vertex function at two loops is $\tilde{\Gamma}_0 = 1$ [4]. The ratio

$$\frac{\tilde{Z}}{\tilde{Z}'} = 1 + C_F T_F \left(\frac{\alpha_s}{4\pi\varepsilon}\right)^2 \left(1 - \tfrac{5}{6}\varepsilon\right) \tag{14}$$

can be obtained from [8]. Finally, we arrive at

$$\tilde{\jmath}(M) = \left[1 + \frac{89}{36} C_F T_F \left(\frac{\alpha_s}{4\pi}\right)^2\right] \tilde{\jmath}'(M)\,. \tag{15}$$

It was stated in [4] that it is impossible to match QCD heavy-light currents to HQET without heavy-quark loops by equating exclusive matrix elements. This statement is incorrect; it was based on eq. (2.1) of that paper, whose right-hand side should be multiplied by $Z_q^{1/2}$ in this case. Taking into account (15), we can write the matching coefficients for HQET without heavy-quark loops in the form (2.20) [4] with $a'_f = a_f + \frac{89}{36}$. Correspondingly, all numerical values of the coefficients of $(\alpha_s/\pi)^2$ in Table 1 [4] should be shifted by 0.103 in this case.

In conclusion, light-light quark currents should be adjusted according to (10) when crossing a heavy-flavour threshold; the heavy-light HQET current (say, $\bar{q}\tilde{b}$) should be adjusted according to (15) when crossing a threshold (say, of $c$ quark).

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