TTP 98-35

Relativistic Positronium Equation

Hartmut Pilkuhn

Institut für Theoretische Teilchenphysik, Universität, D-76128 Karlsruhe, Germany and CERN

(e-mail:hp@particle.physik.uni-karlsruhe.de)

Abstract

The recoil corrections to the energy levels of leptonium are calculated to the order α^6 and $\alpha^6 \log\alpha$ for all states except those having hyperfine mixing. A correction to the hyperfine operator is derived in which the potential appears in the denominator.

PACS number: 36.10.Dr, 03.65.Pm

The calculation of positronium energy levels by closed expressions is a valuable exercise for the handling of relativistic effects in two-body bound state problems. For the hyperfine splitting of the positronium ground state, a closed expression has recently been found for the $m\alpha^6$ "recoil terms" [1]. It agrees with a formula by Pachucki [2], which is valid for all principal quantum numbers n and for arbitrary reduced masses μ, but which still contains an n-independent numerical function $F(\mu)$. The formula has the structure

$$n^3 E^{(6)}/\mu\alpha^6 = f_0(n) + (F(\mu) + f_1(n))\mu^2/m_1 m_2 + f_2(n)(\mu^2/m_1 m_2)^2, \tag{1}$$

where f_0 follows from the Dirac equation for mass μ. The hyperfine-averaged expressions are denoted by $\overline{f}_i$ in the following. For $\overline{f}_2$, Pachucki finds

$$\overline{f}_2 = -2/n - 1/3n^2 - 1/16n^3. \tag{2}$$

In this paper, the n^{-1} and n^{-3} terms of f_2 are given for hyperfine states of total angular momenta $f = 0$ and $f = l \pm 1$, from which the corresponding terms in $\overline{f}_2$ are confirmed. The states with $f = l > 0$ are complicated by hyperfine mixing and are excluded. However, the main point is a modification of the hyperfine operator, which places the potential $V(r) = -\alpha/r$ into the denominator, and which is responsible for the terms of order $\alpha^6 \log \alpha$. Apart from this modificaltion, the equation to be used is the recently derived relativistic two-body equation with eight components [3],

$$(\epsilon - \mu_r \beta - V - \gamma_5 \boldsymbol{\sigma}_1 \boldsymbol{p} - \gamma_5 p_{hf})\psi = 0, \tag{3}$$

$$\mu_r \equiv m_1 m_2 / E, \quad \epsilon \equiv E/2 - (m_1^2 + m_2^2)/2E. \tag{4}$$

From the point of view of perturbative QED, the power of an explicit bound state differential equation depends on its ability to sum Feynman graphs. The present equation sums all the graphs that must be added to other equations for the derivation of (1) [1,2]. The function f_0 in (1) follows by neglecting both the hyperfine operator $\gamma_5 p_{hf}$ and the energy dependence in the denominators of ϵ and μ_r, in which case the equation becomes an ordinary Dirac equation for a particle of reduced mass μ and with $\epsilon = E/2 - (m_1^2 + m_2^2)/2(m_1 + m_2)$. The function f_1 and the last term of f_2 follow from the use of the corresponding Dirac expression to order α^4 for E in the denominators:

$$E = m_1 + m_2 - \mu\alpha^2(1 - 3\alpha^2/4n^2 + \alpha^2/n(j + \tfrac{1}{2}))/2n^2. \tag{5}$$

Technically, the quadratic Dirac equation is used [4]

$$(\epsilon^2 - \mu^2 - 2\epsilon V - \tilde{L}^2/r^2 + \partial_r^2)u = 0, \tag{6}$$

in which the hyperfine operator is converted into an equivalent (to order α^4) r^{-2} -operator and included in an effective centrifugal potential $\tilde{L}^2/r^2$. The eigenvalues $l'(l'+1) = l(l+1) - \alpha^2(1+a)$ of $\tilde{L}^2$ [3] are now rewritten in terms of the combination $\gamma^2 = (j+\frac{1}{2})^2 - \alpha^2$ which appears in the Dirac equation for $l = j - \frac{1}{2}$,

$$l'(l'+1) = \gamma^2 - (\gamma^2 + \alpha^2 a_{hf})^{1/2}, \tag{7}$$

where a_{hf} is only the hyperfine part of the parameter a. For $l = 0$, a_{hf} is $2\epsilon/E$ and $-2\epsilon/3E$ in the singlet and triplet states, respectively. The expectation value of the remaining operators vanishes to order α^4. The first term in $\overline{f}_2$ arises from averaging over $(\delta_l)^2$ (see the expression for b in equation (39) of [3]), while the last one is independent of l, j, and f.

The main point of this paper is the modification of the previous hyperfine operator [3] by a factor $f_-^{-1} = (1 - 2V/E)^{-1}$:

$$p_{hf} = -i\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 V(E - 2V)^{-1}\boldsymbol{p}. \tag{8}$$

To see its necessity, consider the operators on the left-hand side of (3) for $m_1 = m_2$, namely $\epsilon + \mu - V = (E^2 - 4m_1^2)/2E - V$ in the large Dirac components, and $\epsilon - \mu - V = E/2 - V \equiv f_- E/2$ in the small ones. Elimination of the small Dirac components leads to the following equation for the large ones, called ψ_g:

$$(E^2/4 - m_1^2 - \tfrac{1}{2}EV)\psi_g = p_{hf} f_-^{-1} p_{hf}\psi_g, \quad f_- = 1 - 2V/E. \tag{9}$$

For parapositronium, the relevant position of the unmodified operator $p_{hf,un}$ to the right of f_-^{-1} begins with f_- (see the upper right corner of equation (30) in [3], for $F = 0$), which is thus cancelled out. The relevant position of $p_{hf,un}$ to the left of f_-^{-1} begins with f_+, which may be divided off. The resulting differential equation is reduced to the confluent hypergeometric one, in a shifted variable $r' = r - 2\alpha/E$. Its singularity at $r' = 0$ occurs at $r = 2\alpha/E$, which excludes any bound state interpretation. The additional factor f^{-1} of (8) is necessary to shift the singularity back to $r = 0$. For $l = 0$, it produces the logarithmic energy shift

$$E_{log}^{(6)} = -(8n^3)^{-1} m_1 \alpha^6 \log(2\alpha), \tag{10}$$

in agreement with previous calculations. For $l > 0$, f_-^{-1} may be expanded as follows:

$$f_-^{-1} = (1 - 2V/E)^{-1} = 1 + 2V/E + 4V^2/E^2. \tag{11}$$

Insertion of the quadratic term into (9) produces the operator $\boldsymbol{\sigma}_1 \boldsymbol{p} V^2 \boldsymbol{\sigma}_1 \boldsymbol{p}/E^2$, the angular part of which confirms the $V^2 l(l+1)/r^2$ potential of Yelkhovsky [5].

The operator $\gamma_5 p_{hf}$ in (3) arises from the elimination of components. Its new form may be related to the Fourier transform of the Born graph by replacing $e^{i\boldsymbol{qr}}$ by $e^{i\boldsymbol{qr}'}$, with $\boldsymbol{r}' = \boldsymbol{r}(1 + 2\alpha/Er)$. A perturbative justification of this modification could be possible in combination with gauge invariance. The combination $E - 2V$ in (8) happens to be the same as in the main term $\epsilon - V$ in (3). After multiplication by E and introduction of a new variable $\rho = r/E$, one sees that E occurs only in the combination

$$s_V = E^2 - 2V(\rho). \tag{12}$$

And as $E^2 = (p_1 + p_2)^2$ is a Lorentz invariant, (12) is simpler than the traditional $E - V(r)$ which contains square roots of Lorentz invariants.

It may be allowed to add a speculation on the hyperfine splitting between vector (1^-) and pseudoscalar (0^-) mesons in quarkonium potential models. The splitting in E^2, $\Delta = m^2(1^-) - m^2(0^-)$, agrees with a perturbative treatment in heavy quarkonium, where it is small and relatively constant [6,7]. But it increases slightly with decreasing quark masses and becomes the dominant effect for the light pseudoscalar and vector meson octets, leading to $\Delta \to m^2(1^-)$ as $m^2(0^-) \to 0$. An explanation of this strange dominance of the hyperfine operator could be as follows: Whereas the perturbative QCD potential V is kept in the numerator of the hyperfine operator, a nonperturbative potential $W(\rho)$ could appear in the combination $s_W = E^2 - 2W(\rho)$. If now $s_W(\rho \to \infty) = 0$, then the hyperfine interaction would remain large even for $V(\rho \to \infty) = 0$. And as lighter systems have larger radii in potential models, their hyperfine splitting would be expexted to increase.

The author would like to thank Dr. J.O. Eeg for the kind hospitality extended to him at the physics institute of the University of Oslo, where this work was begun, Drs. Th. Mannel, K. Melnikov, and K. Pachucki for helpful comments, and the Volkswagenstiftung for financial support of his sabbatical leave of absence from Karlsruhe.

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