THE PERTURBATIVE QCD POTENTIAL AND THE $t\bar{t}$ THRESHOLD*

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We include the full second-order corrections to the static QCD potential in the analysis of the $t\bar{t}$ threshold cross section. There is an unexpectedly large difference between the QCD potential improved by the renormalization-group equation in momentum space and the potential improved by the renormalization-group equation in coordinate space. This difference remains even at a fairly short distance $1/r \simeq 100$ GeV and its origin can be understood within perturbative QCD. We scrutinize the theoretical uncertainties of the QCD potential in relation to the $t\bar{t}$ threshold cross section. In particular there exists a theoretical uncertainty which limits our present theoretical accuracy of the $t\bar{t}$ threshold cross section at the peak to be $\delta\sigma_{\rm peak}/\sigma_{\rm peak} \gtrsim 6\%$ within perturbative QCD.

In this paper, based on Ref. [1], we report on our present theoretical understanding of the $t\bar{t}$ total cross section near the threshold. Up to now, all the $\mathcal{O}(\alpha_s)$ corrections (also leading logarithms) have been included in the calculations of various cross sections near threshold. In order to take into account the QCD binding effects properly in the cross sections, we have to systematically rearrange the perturbative expansion near threshold. Namely, we first resum all the leading Coulomb singularities $\sim (\alpha_s/\beta)^n$, take the result as the leading order contribution, and then calculate higher order corrections, which are essentially resummations of the terms $\sim \alpha_s^{n+1}/\beta^n$, α_s^{n+2}/β^n, ... It is also important to resum large logarithms arising from the large scale difference involved in the calculation [2]. This is achieved by (first) calculating the Green function of the nonrelativistic Schrödinger equation with the QCD potential [3,2]. Conventionally both the coordinate-space approach developed in Refs. [2,4] and the momentum-space approach developed in Refs. [5,6] have been used in solving the equation by different groups independently. It has recently been found [7] that there are discrepancies in the results obtained from the two approaches reflecting the difference in the construction of the potentials in both spaces. It was argued that the differences are formally of $\mathcal{O}(\alpha_s^2)$ but their size turns out to be non-negligible.

Quite recently there has been considerable progress in the theoretical calculations of the second-order corrections to the cross section at threshold and the Coulombic bound-state problem. New contributions have been calculated analytically [8,9] and numerically [10] for QED bound-states. Very important steps have been accomplished in QCD as well. The full second-order correction to the static QCD potential was computed in [11]. Also, the $\mathcal{O}(\alpha_s^2)$ total cross section is known now [12] in the region $\alpha_s \ll \beta \ll 1$ as a series expansion in β. All these results have to be included in the calculation of the full $\mathcal{O}(\alpha_s^2)$ corrections to the threshold cross section, which has just been completed (as far as the production process of top quarks are concerned) [13]. The full second-order corrections turned out to be anomalously large, which may suggest a poor convergence of the perturbative QCD in the $t\bar{t}$ threshold region.

In this paper, we incorporate the full $\mathcal{O}(\alpha_s^3)$ corrections (the second-order corrections to the leading contribution) to the static QCD potential into our analyses. In principle this is a step towards an improvement of the theoretical precision in our analysis of the $t\bar{t}$ threshold cross section. Then we scrutinize the problem of the difference between the momentum-space and the coordinate-space potentials. Contrary to our expectation, the inclusion of the above corrections does not reduce the difference of the cross sections significantly, and there still remains a non-negligible deviation. We find that there is a theoretical uncertainty within perturbative QCD which limits our present-day theoretical accuracy of the threshold cross section.

Let us now briefly explain the construction of our potentials in momentum space and in coordinate space, respectively. More detailed descriptions including formulas can be found in [1]. The large-momentum part of the momentum-space potential $V_{\rm JKPT}(q)$ is determined as follows. First the potential has been calculated up to $\mathcal{O}(\alpha_s^3)$ in a fixed-order calculation. The result is then improved using the three-loop renormalization group equation in momentum space. At low momentum, the potential is continued smoothly to a Richardson-like potential. On the other hand, the short-distance part of the

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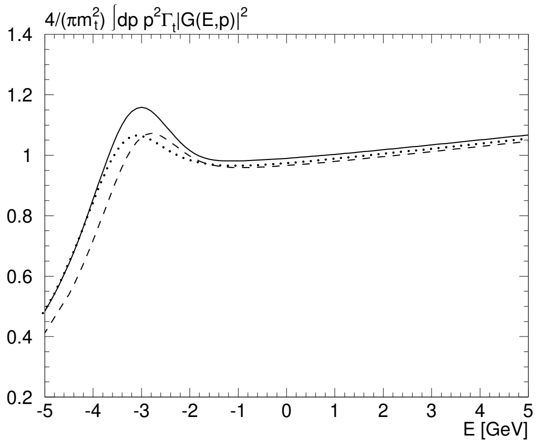


Figure 1: Comparison of the total cross sections (normalized to R) calculated from the different potentials: $V_{\rm JKPT}$ (solid), $V_{\rm SFHMN}$ (dashed), and $V_{\rm new}$ (dotted line). We set $\alpha_{\overline{\rm MS}}(M_Z^2) = 0.118$, $m_t = 175$ GeV, and $\Gamma_t = 1.421$ GeV.

coordinate-space potential $V_{\rm SFHMN}(r)$ is calculated by taking the Fourier transform of the fixed-order perturbative potential in momentum space, and then is improved using the three-loop renormalization group equation in coordinate space. At long distance, the potential is continued smoothly to a phenomenological ansatz. Thus, it is important to note that the two potentials are *not* the Fourier transforms of each other even in the large-momentum or short-distance region. They agree only up to the next-to-next-to-leading logarithmic terms of the series expansion in a fixed $\overline{\rm MS}$ coupling. The difference begins with the non-logarithmic term in the three-loop fixed-order correction.

In Fig. 1 we show a comparison of the total cross sections (normalized to R) calculated from $V_{\rm JKPT}$ (solid) and from $V_{\rm SFHMN}$ (dashed line), without any weak or hard-gluon corrections:

$$R = \frac{4}{\pi m_t^2} \int_0^\infty dp\, p^2 |G(E,p)|^2\, \Gamma_t. \tag{1}$$

For the physical parameters we used $\alpha_{\overline{\rm MS}}(M_Z^2) = 0.118$, $m_t = 175$ GeV, and $\Gamma_t = 1.421$ GeV. We find that the two cross sections differ by 7.8% at the peaks and by 2.2% at $E = 5$ GeV.[a] Since the difference of the cross sections calculated from the next-to-leading order potentials is 8.6% at the peak and 2.4% at $E = 5$ GeV for the same value of $\alpha_s(M_Z^2)$, the cross sections have come closer only slightly after the inclusion of the second-order correction to the potential. The remaining difference is much larger than what one would expect from an $\mathcal{O}(\alpha_s^3)$ correction relative to the leading order, which is not fully included in our analyses, even if we take into account the high sensitivity to the coupling [2], $\sigma_{\rm peak} \propto \alpha_s^2$. The purpose of this paper is to understand the origin of this unexpectedly large difference.

In order to compare the asymptotic behavior of the potentials more clearly, we plot in Fig. 2(a) the coordinate-space effective couplings defined by

$$\bar{\alpha}_{\rm V}(1/r) = \left(-C_F/r\right)^{-1} V(r) \tag{2}$$

for $V_{\rm JKPT}(r)$ and $V_{\rm SFHMN}(r)$ as solid and dashed lines, respectively. Contrary to our expectation, the difference of the couplings exceeds 3% even at very short distances, $1/r \simeq 100$ GeV.

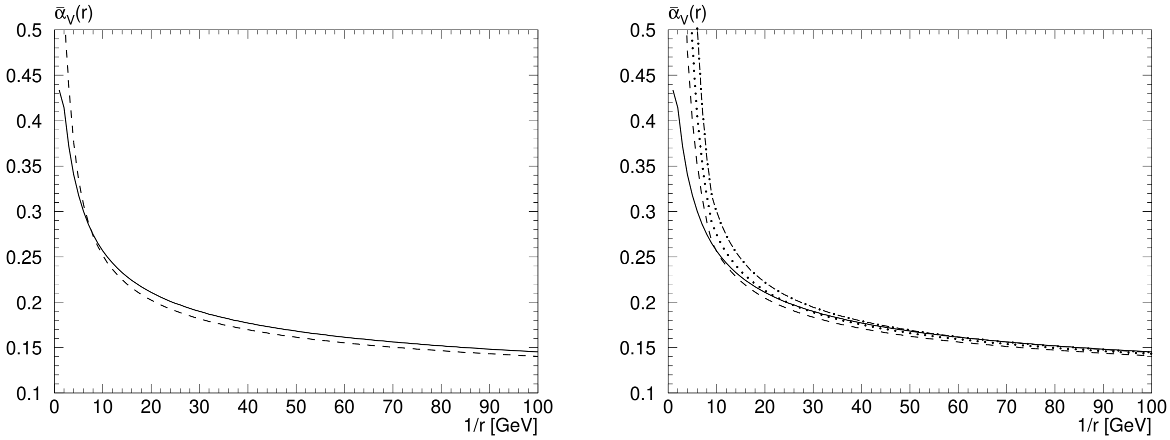
Naturally the question arises: Why is there such a large discrepancy between the potential constructed in momentum space and that constructed in coordinate space? To answer this question, let us examine a relation connecting the effective coupling in coordinate space, defined by Eq. (2), and the effective coupling in momentum space, defined from the momentum-space potential as

$$\alpha_{\rm V}(q) = \left(-4\pi C_F/q^2\right)^{-1} V(q). \tag{3}$$

The relation is derived from the renormalization group equation of $\alpha_{\rm V}(q)$ and exact to all orders. In the asymptotic region where the couplings are small, it can be given in the form of an asymptotic series [14], which reads numerically

$$\bar{\alpha}_{\rm V}(1/r) = \alpha_{\rm V} + 1.225\,\alpha_{\rm V}^3 + 5.596\,\alpha_{\rm V}^4 + 32.202\,\alpha_{\rm V}^5 + \ldots \tag{4}$$

[a] In this paper we are not concerned with those differences of the cross sections which can be absorbed into an additive constant to the potential $V(r)$, or equivalently, into a redefinition of the top quark mass. The theoretical uncertainty in the pole mass for our problem is discussed in [14].



(a) (b)

Figure 2: (a) Comparison of the coordinate-space effective charges defined from $V_{\rm JKPT}$ (solid) and from $V_{\rm SFHMN}$ (dashed line). (b) Comparison of the coordinate-space effective charges defined from the various terms of Eq. (4). See the text for the description of each curve.

for $n_f = 5$. On the right-hand-side, $\alpha_{\rm V} = \alpha_{\rm V}(q = e^{-\gamma_E}/r)$. All terms which are written explicitly are determined from the known coefficients of the β function, β_0^V, β_1^V, and β_2^V. At present, we can use the above relation consistently only at $O(\alpha_{\rm V}^3)$ because we know the effective couplings only up to the next-to-next-to-leading order corrections in perturbative QCD, i.e. we know the relation between $\alpha_{\rm V}$ and $\alpha_{\overline{\rm MS}}$ only up to $\mathcal{O}(\alpha_{\overline{\rm MS}}^3)$. Due to this limitation, essentially, the effective coupling $\bar{\alpha}_{\rm V}$ defined from $V_{\rm SFHMN}$ is the right-hand-side of the above equation truncated at the $O(\alpha_{\rm V}^3)$ term, while $\bar{\alpha}_{\rm V}$ defined from $V_{\rm JKPT}$ is the right-hand-side including all terms. Numerically, the $O(\alpha_{\rm V}^4)$ term and the $O(\alpha_{\rm V}^5)$ term contribute as $+1.4\%$ and $+1.1\%$ corrections, respectively, for $\alpha_{\rm V} = 0.1379$ (corresponding to $1/r = 100$ GeV). Therefore, these higher order terms indeed explain the difference of the effective couplings at small r. Fig. 2(b) shows several curves derived from the above relation:

1. The solid line is $\bar{\alpha}_{\rm V}(1/r)$ defined from $V_{\rm JKPT}$.

2. The dashed curve is $\alpha_{\rm V} + 1.225\,\alpha_{\rm V}^3$, where $\alpha_{\rm V} = \alpha_{\rm V}(q = e^{-\gamma_E}/r)$ is calculated using the perturbative prediction in momentum space. This curve is essentially the same as $\bar{\alpha}_{\rm V}(1/r)$ defined from $V_{\rm SFHMN}(r)$, since it is the next-to-next-to-leading order perturbative prediction for the coordinate-space coupling at short distances.

3. The dotted curve includes the next correction, $5.596\,\alpha_{\rm V}^4$, which is in fact even larger than the $\mathcal{O}(\alpha_{\rm V}^3)$ term below $1/r \sim 30$ GeV.

4. The dash-dotted curve includes the $\mathcal{O}(\alpha_{\rm V}^5)$ term.

We observe that the agreement of both sides of Eq. (4) becomes better as we include more terms at small r, while it becomes worse at large r on account of the asymptoticness of the series. From the purely perturbative point of view, the discrepancy between our two potentials, $V_{\rm JKPT}$ and $V_{\rm SFHMN}$, in the asymptotic region thus seems real, an indication of large higher order corrections. When the third-order correction to the potential will be computed in terms of $\alpha_{\overline{\rm MS}}$ in the future, the $\mathcal{O}(\alpha_{\rm V}^4)$ term will be treated consistently and the difference will reduce by 1.4% at $1/r \simeq 100$ GeV.

We may consider this difference of $\bar{\alpha}_{\rm V}(1/r)$ at short-distances as an estimate of the higher order corrections on the basis of the following observations. First, the two potentials are equal up to the next-to-next-to-leading order, and there seems to be no reason *a priori* for considering one of the two to be more favorable theoretically. Secondly, if we apply the same method (the relation between $\bar{\alpha}_{\rm V}$ and $\alpha_{\rm V}$) to estimate the size of the already known $\mathcal{O}(\alpha_s^3)$ correction, we obtain $\pi^2\beta_0^2/3 = 193.4$, which turns out to be a slight under-estimate of the true correction [11] $a_2 = 333.5$ ($n_f = 5$).

Moreover, the above 3% uncertainty of $\bar{\alpha}_{\rm V}(1/r)$ at $1/r \simeq 100$ GeV provides a certain criterion for the present theoretical uncertainty of the $t\bar{t}$ cross section. In fact, it would already limit the theoretical accuracy of $\bar{\alpha}_{\rm V}(1/r)$ at longer distances to be not better than 3%. If we combine this with a naive estimate $\sigma_{\rm peak} \propto \bar{\alpha}_{\rm V}^2$, we expect a theoretical uncertainty of the peak cross section to be $\delta\sigma_{\rm peak}/\sigma_{\rm peak} \gtrsim 6\%$. Therefore, the large discrepancy of the cross sections which we have seen turns out to be quite consistent with this estimated uncertainty.

One is tempted to include one more term of the series (4) to define a (new) coordinate-space potential despite our ignorance of the corresponding terms in the relation between $\alpha_{\rm V}$ and $\alpha_{\overline{\rm MS}}$, since this would apparently reduce the difference between the two effective couplings. In fact we did this exercise, but (to our surprise) it did not bring the cross section closer to the one calculated from the momentum-space potential $V_{\rm JKPT}$ in the peak region. This cross section calculated from the potential $V_{\rm new}(r)$, which incorporates the $\mathcal{O}(\alpha_{\rm V}^4)$ term of Eq. (4), is shown as a dotted curve in Fig. 1. This fact indicates that we are no longer able to improve the agreement of the cross sections by including even higher order terms, as we are confronting the problem of asymptoticness of the series.

Some indications can be obtained by looking into the nature of the perturbative expansion of each potential. Within our present knowledge of the static QCD potential, the perturbative series looks more convergent for the momentum-space potential than for the coordinate-space potential. To see this, one may compare the β functions of the effective couplings (the V-scheme couplings) in both spaces [11]. Numerically, the first three terms in the perturbative expansion read

- (momentum-space coupling)

$$\mu^2 \frac{d\alpha_{\rm V}}{d\mu^2} = -0.6101\,\alpha_{\rm V}^2 - 0.2449\,\alpha_{\rm V}^3 - 1.198\,\alpha_{\rm V}^4 + \dots \tag{5}$$

- (coordinate-space coupling)

$$\mu^2 \frac{d\bar{\alpha}_{\rm V}}{d\mu^2} = -0.6101\,\bar{\alpha}_{\rm V}^2 - 0.2449\,\bar{\alpha}_{\rm V}^3 - 1.945\,\bar{\alpha}_{\rm V}^4 + \dots \tag{6}$$

for $n_f = 5$. The first two coefficients are universal. The third coefficient depends on the scheme (the definition) of the coupling. As the third coefficients for the V-scheme couplings are quite large, the third term of the β function is comparable to the second term already for $\alpha_V = 0.20$ and for $\bar{\alpha}_V = 0.13$, respectively. The difference of the third coefficients between momentum space and coordinate space originates from the $\pi^2\beta_0^2/3$ term. Although the magnitude of the third coefficients is of the same order, in practice it makes a certain difference whether an apparent convergence is lost at $\alpha_V = 0.20$ or $\bar{\alpha}_V = 0.13$ because there is a large scale difference between the two values. This indicates a worse convergence in coordinate space than in momentum space.

It is interesting to examine the level of uncertainties within the momentum-space approach or the coordinate-space approach by itself. Fig. 3 shows how the cross section changes when we vary the scale by a factor of 2 in each approach: from $\mu = q/\sqrt{2}$ to $\mu = \sqrt{2}q$ in Eq. (28) of [11] in the momentum-space approach (upper three curves), and from $\mu = \mu_2/\sqrt{2}$ to $\mu = \sqrt{2}\mu_2$ in Eq. (44) of [11] in the coordinate-space approach (lower three curves). For the momentum-space approach, the variation of the cross section is 2.2% at the peak and around 0.6% for c.m. energies above threshold. Meanwhile in the coordinate-space approach, the variation of the cross section amounts to 0.7% at the peak and 0.9% at larger c.m. energies. These results may be regarded as an internal consistency check for each approach.

So far we have examined the difference between the momentum-space potential and the coordinate-space potential in detail, and we have taken this difference as an estimate of the theoretical uncertainty of the QCD potential. This estimation may, however, be somewhat misleading. One might well argue that the difference is an artifact of our inadequate use of the perturbative expansion in describing the potentials related by Fourier transformation. For illustration, let us consider a hypothetical case where we know $\alpha_{\rm V}(q)$ exactly. In this case, if we calculate $\bar{\alpha}_{\rm V}(1/r)$ via (numerical) Fourier transformation of the momentum-space potential, in principle we can calculate $\bar{\alpha}_{\rm V}(1/r)$ to any desired accuracy by investing more time. On the other hand, if we calculate $\bar{\alpha}_{\rm V}(1/r)$ using the series on the right-hand-side of Eq. (4) naively, there is a limitation in the achievable accuracy because the series is only asymptotic. Certainly such a limited accuracy does not reflect any theoretical uncertainty of $\bar{\alpha}_{\rm V}(1/r)$. This nature should not be confused with our claim. We claim that there is a limitation within perturbative QCD in relating $\alpha_{\rm V}(q)$ or $\bar{\alpha}_{\rm V}(1/r)$ to $\alpha_{\overline{\rm MS}}(M_Z^2)$, and estimate theoretical uncertainties in this relation using the difference of the potentials in the two spaces.

Let us comment on the relation between this work and the work [13] which has been completed very recently. Ref. [13] presents a fixed-order calculation (without log resummations), which includes the full second-order corrections to the $t\bar{t}$ threshold cross sections. While we include corresponding corrections only to the static QCD potential in our analyses, we also employ the renormalization-group improvement and thus resum large logarithms in the potential. In this sense the two works are complementary to each other. Both works give the common conclusion that the second-order corrections to the $t\bar{t}$ threshold cross section are large and may indicate a poor convergence of perturbative QCD, although they are based on qualitatively different arguments. The full set of the fixed-order $\mathcal{O}(\alpha_s^2)$ corrections to the cross section near threshold [13] are larger in size and even more scale dependent than the corrections to the potential alone. The theoretical uncertainty may therefore even be larger than indicated by the study in our paper.

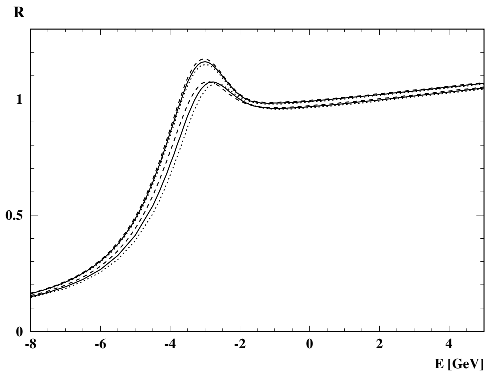


Figure 3: Comparison of the cross sections for different choices of the scale. The upper three curves are for the momentum-space approach: $\mu = q$ (solid), $\mu = \sqrt{2}q$ (dotted), and $\mu = q/\sqrt{2}$ (dashed line). The lower three curve are for the coordinate-space approach: $\mu = \mu_2 = \exp(-\gamma_E)/r$ (solid), $\mu = \sqrt{2}\mu_2$ (dotted), and $\mu = \mu_2/\sqrt{2}$ (dashed line).

Summary

- There is a difference between the potential constructed in momentum space and that constructed in coordinate space even at a fairly short-distance, $1/r \sim 100$ GeV. The difference can be understood within the framework of perturbative QCD. We already know that there is a large correction at $\mathcal{O}(\alpha_s^4)$ in the relation between the two potentials, although a consistent treatment is not possible until the full $\mathcal{O}(\alpha_s^4)$ corrections to the QCD potential are calculated.

- The above difference at short distances would limit the theoretical accuracy of the QCD potential at longer distances and thus provides a criterion for our present theoretical uncertainty of the $t\bar{t}$ cross section, $\delta\sigma_{\rm peak}/\sigma_{\rm peak} \gtrsim 6\%$.

- In addition, it seems that we are confronting the problem of the asymptoticness of the perturbative series in the calculation of the $t\bar{t}$ cross section, as the top quarks do not probe a region which is sufficiently deep in the potential. We may not be able to improve our theoretical precision even if the higher order corrections are calculated in perturbative QCD.

- We may, however, discuss which of the two approaches gives a more favorable result theoretically. Up to the second-order corrections, the perturbative series looks better convergent for the momentum-space potential than for the coordinate-space potential.

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