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Determining the Strange Quark Mass in Cabibbo Suppressed Tau Lepton Decays

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Abstract

In this work radiative corrections in the total hadronic decay rate of the τ lepton and some moments of its differential distributions are studied employing perturbative QCD and the operator product expansion. We calculate quadratic quark mass corrections in the strange mass to the decay rate ratio R_τ to the order $\mathcal{O}(\alpha_s^3 m^2)$ and find that they contribute appreciably to the Cabibbo suppressed decay modes of the τ-lepton. Using the results of a recent experimental analysis, we obtain $m_s(1\text{ GeV}) = 200 \pm 40_{exp} \pm 30_{th}$ MeV.

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1 Introduction

With an ever increasing number of τ leptons observed by the four LEP experiments and by CLEO the Cabibbo suppressed τ decays have become one of the important topics of recent experimental analysis [1]. The study of exclusive channels allows to determine hadronic resonance parameters, to test the predictions in the chiral limit and, quite generally, to explore the hadronic current in the low energy region. Multi-differential angular distributions of mesons can be used to measure the polarization of the τ lepton and to furthermore disentangle the various spin parity contributions of hadronic states with $J^P = 0^+$, 0^-, 1^- or 1^+ induced by the (non-) conserved parts of vector and axial vector current respectively [2]. Complementary studies are based on the inclusive decay rate derived from the semileptonic branching ratio or the τ life time (see e.g. [3, 4, 5, 6, 7]). The determination of the strong coupling constant α_s has been a focal point in these investigations. The small reduction of the Cabibbo suppressed rate (relative to the massless prediction) has been used recently [8] even to extract a value for the strange quark mass. The analysis was based on the total rate and a theoretical calculation [9] including quadratic mass terms with coefficients of order α_s^2. Recently, the authors of [9] have revised their analysis and corrected an error in the published numerical value of the α_s^2 term. As a result the contribution from that term has increased dramatically by a factor of about four [10, 11]. This new development calls for a fresh look at the possibility for an m_s determination from τ decays.

In this work the theoretical analysis will be extended and improved in various ways:

- QCD corrections to the mass terms of order α_s^3 will be calculated for some of the moments of the spectral functions which if included lead to a sizeable shift of the predictions.

- It is proposed to use the techniques of [2] to separate the states of different spin parity, allowing thus for quark mass determinations from four independent spectral functions. Eventually even different moments of their respective spectral functions might be considered, leading to additional tests of the method.

- The method [12] of resummation of effects from the running of both the coupling constant and the strange quark mass along the contour of integration in the complex plane through the renormalization group improvement is used for τ lepton observables to provide better convergence of the perturbative series.

The configurations with 0^+ and 0^- and non-vanishing invariant masses are strictly forbidden in the limit of massless quarks. The integrated rate, i.e. the lowest moment of the spin separated distribution exhibits a remarkable dependence on the perturbative mass of the strange quark and a nonperturbative parameter at the same time. Higher moments are free from this nonperturbative constant, allowing thus, at least in principle, a new determination of the strange quark mass.

The paper is organised as follows. In the next section some general relations are given and our notation is fixed. The observables to be analysed in τ lepton physics are introduced and the stage is set for their perturbative analysis. In Sect. 3 some new features are described which appear due to the mass terms in the Cabibbo suppressed channel. Explicit expressions for the coefficients of quadratic mass terms are given. In the same section both the finite order and resumed observables are presented. Sect. 4 concentrates on nonperturbative corrections to spin zero contribution. In sect. 5 numerical results are given. The last section 6 contains our conclusion about the possibility and accuracy of a strange quark mass determination from an inclusive analysis of Cabibbo suppressed modes.

2 Generalities

In order to set the framework of the subsequent discussion, which includes quark mass effects and the separation of the spin one and spin zero contributions, we repeat below a number of generalities about the theoretical analysis of τ lepton observables both in massless and massive case and also introduce our notations for necessary quantities which can be found in earlier literature (see, e.g. Refs. [7, 13]).

2.1 Correlators

Physical τ lepton observables are related to correlators of vector and axial vector currents of light quarks that are defined as follows

$$\begin{aligned} \Pi^{V/A}_{\mu\nu,ij}(q,m_i,m_j,m,\mu,\alpha_s) &= i\int dx e^{iqx}\langle T[\, j^{V/A}_{\mu,ij}(x)(j^{V/A}_{\nu,ij})^\dagger(0)\,]\rangle \\ &= g_{\mu\nu}\Pi^{[1]}_{ij,V/A}(q^2)+q_\mu q_\nu \Pi^{[2]}_{ij,V/A}(q^2) \end{aligned} \tag{1}$$

with $m^2=\sum_{f=u,d,s} m_f^2$ and $j^{V/A}_{\mu,ij}=\bar{q}_i\gamma_\mu(\gamma_5)q_j$. Here q_i and q_j are two (generically different) quarks with masses m_i and m_j respectively. In the present paper we work within QCD with effective three light quarks and do not consider corrections due to heavy quarks (c-quark) that can enter in higher orders of PT through internal loops [14].

An important and convenient property for the analytic analysis of the above introduced polarization functions $\Pi^{[1,2]}_{ij,V/A}(q^2)$ is the absence of the so-called kinematical singularities because no additional factors of momenta appear in the defining Eq. (1). For these polarization functions the dispersion relations are valid that describe the physical states contributing to the correlator (1)

$$\Pi^{[l]}_{ij,V/A}(q^2) = \frac{1}{12\pi^2}\int_{s_0}^{\infty} ds \frac{(-s)^{2-l} R^{(l)}_{ij,V/A}(s, m_u, m_d, m, \mu, \alpha_s)}{s - q^2} \qquad \text{mod sub} \tag{2}$$

where $l = 1, 2$, and the proper powers of s are introduced in the definition of $R(s)$ to make the spectral densities positive and dimensionless. In perturbation theory, the threshold is at $s_0 = (M_i + M_j)^2$ (where M_i denotes the pole mass of a quark) while the true thresholds are, say, $4m_\pi^2$ for $\Pi^{[1]}_{ud}$ and m_π^2 for $\Pi^{[2]}_{ud}$. It should be noted that the spectral density $R^{(2)}$ contains contributions from spin one as well as from spin zero intermediate states. The spectral density $R^{(0)}$ that is free from contributions of hadronic states with spin 1 is defined by

$$R^{(0)} \equiv R^{(2)} - R^{(1)}.$$

Another useful representation of the tensor $\Pi^{V/A}_{\mu\nu,ij}(q)$ in terms of scalar functions reads

$$\Pi^{V/A}_{\mu\nu,ij}(q, m_i, m_j, m, \mu, \alpha_s) = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi^{(1)}_{ij,V/A}(q^2) + q_\mu q_\nu \Pi^{(0)}_{ij,V/A}(q^2) \tag{3}$$

where the correlator is decomposed into the components $\Pi^{(0,1)}_{ij,V/A}(q^2)$ that contains contributions of the states with the angular momentum $J = 0$ and $J = 1$ respectively.

A direct comparison of (1) and (3) leads us to the following relations

$$\Pi^{(1)} = -\Pi^{[1]}/q^2, \quad \Pi^{(0)} = \Pi^{[2]} + \Pi^{[1]}/q^2. \tag{4}$$

In general $\Pi^{[1]}(0)$ may be different from zero which implies a kinematical singularity (pole) in both $\Pi^{(1)}$ and $\Pi^{(0)}$ that describe the spin structure of the correlator (1).

2.2 Ward identity

The divergence of the (axial)vector current is known through equations of motion for the fields in QCD and is given by (pseudo)scalar two-quark operators in the case of massive quarks. In the massless limit both (nonsinglet) axial and vector currents are conserved. Nevertheless their behaviour is different and, for the axial current, is governed by the spontaneous violation of chiral symmetry and the existence of the massless excitation accompanying this violation – the Goldstone boson, in our case the pion or the kaon. The (axial)vector and (pseudo)scalar correlators are connected through a Ward identity

$$q_\mu q_\nu \Pi^{\mathrm{V/A}}_{\mu\nu,ij}(q) = (m_i \mp m_j)^2 \Pi^{\mathrm{S/P}}_{ij}(q) + (m_i \mp m_j)(\langle \overline{\psi}_{\mathrm{i}} \psi_{\mathrm{i}} \rangle \mp \langle \overline{\psi}_{\mathrm{j}} \psi_{\mathrm{j}} \rangle) \tag{5}$$

where

$$i \int dx e^{iqx} \langle T[\, j^{S/P}_{ij}(x) (j^{S/P}_{ij})^\dagger(0)\,] \rangle = \Pi^{S/P}_{i,j}(q) = Q^2 \Pi^{(S/P)}_{i,j}(q) \tag{6}$$

and $j^{S/P}_{ij} = \bar{q}_i (i\gamma_5) q_j$, $Q^2 = -q^2$. The dimensionless function $\Pi^{(S/P)}_{i,j}(q)$ is related to the correlator of (pseudo)scalar currents $\Pi^{S/P}_{i,j}(q)$ by one power of Q^2.

2.3 Contour integrals and τ lepton observables

The hadronic decay rate of the τ lepton is obtained by integrating the absorptive parts of the spectral functions with respect to the invariant hadronic mass. Corresponding to two different tensor decompositions (1), (3) two different integral representations can be obtained. The first one displays the structure of hadronic contributions classified according to their spin

$$\begin{aligned} R_\tau = &\ R_\tau^{(1)} + R_\tau^{(0)} \\ = &\ \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{M_\tau^2}\right) R^{(1)}(s) + R^{(0)}(s)\right] \end{aligned} \tag{7}$$

where

$$R^{(J)} = |V_{ud}|^2 (R^{(J)}_{ud,V} + R^{(J)}_{ud,A}) + |V_{us}|^2 (R^{(J)}_{us,V} + R^{(J)}_{us,A}), \quad J = 0, 1. \tag{8}$$

The representation of the total decay rate through the absorptive parts of the structure functions $\Pi^{[1]}$ and $\Pi^{[2]}$ is simpler from the point of view of its analytic properties for continuation into the complex plane and reads

$$R_\tau = \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\frac{2s}{M_\tau^2} R^{(1)}(s) + R^{(2)}(s)\right]. \tag{9}$$

Due to the analyticity of $\Pi^{[1,2]}$ in the cut complex s-plane (the absence of singularities away from the physical cut, even of kinematical singularities at the origin) R_τ can be expressed as the contour integral along a circle C of the radius $|s| = M_\tau^2$

$$R_\tau = 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\Pi^{[2]}(s) - \frac{2}{M_\tau^2} \Pi^{[1]}(s)\right]. \tag{10}$$

As the behaviour of $\Pi^{[1,2]}(s)$ along the "the large circle" of radius $|s| = M_\tau^2$ is assumed to be reliably evaluated within pQCD, representation (10) leads to a well-defined pQCD prediction for R_τ. Unfortunately, this is not true for the spin-separated parts. Indeed, the direct use of (4) leads to

$$
\begin{aligned}
R_\tau^{(1)} &= 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^{(1)}(s)+\Pi^{[1]}(0)/s\right], \\
R_\tau^{(0)} &= 6i\pi \int_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\Pi^{(0)}(s)-\Pi^{[1]}(0)/s\right].
\end{aligned}
\tag{11}
$$

where the contribution of the singularity at the origin (proportional to $\Pi^{[1]}(0)$) has to be included. A nonvanishing value of $\Pi^{[1]}(0)$ is certainly a nonperturbative constant. Thus, within pQCD we cannot predict the decay rates $R_\tau^{(1,0)}$ separately. In the massless limit $\Pi^{(0)}=0$ within perturbation theory and $R^{(0)}$ is saturated by $\Pi^{[1]}(0)$ corresponding to the massless pion (kaon) pole for the axial part of the correlator.

On the other hand, the unknown constant drops out if one considers moments

$$
R_\tau^{(1,0)k,l}(s_0) = \int_0^{s_0} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^k \left(\frac{s}{M_\tau^2}\right)^l \frac{dR_\tau^{(1,0)}}{ds}, \tag{12}
$$

with $k\geq 0,\;\; l\geq 1$. (Note that the moments introduced in [15] are related to ours as $R_\tau^{kl} = R_\tau^{(1)k,l} + R_\tau^{(0)k,l}$.)

The decay rate R_τ may be expressed as the sum of different contributions corresponding to Cabibbo suppressed or allowed decay modes, vector or axial vector contributions and the mass dimension of the corrections

$$
R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S} \tag{13}
$$

with

$$
\begin{aligned}
R_V &= \frac{3}{2}|V_{ud}|^2\left(1+\delta_0+\sum_{D=2,4,\ldots}\delta_{V,ud,D}\right), \\
R_A &= \frac{3}{2}|V_{ud}|^2\left(1+\delta_0+\sum_{D=2,4,\ldots}\delta_{A,ud,D}\right), \\
R_S &= 3|V_{us}|^2\left(1+\delta_0+\sum_{D=2,4,\ldots}\delta_{us,D}\right).
\end{aligned}
\tag{14}
$$

Here D indicates the mass dimension of the fractional corrections $\delta_{V/A,ij,D}$, and $\delta_{ij,D}$ denotes the average of the vector and the axial vector contributions: $\delta_{ij,D} = (\delta_{V,ij,D}+\delta_{A,ij,D})/2$. If a decomposition into different spin/parity contributions is made or a particular pattern of moments is considered then we will use the corresponding obvious generalization of (14). For instance,

$$
R_{S,V}^{(1)kl} = a_{kl}\,|V_{us}|^2\left(1+\delta_0^{kl}+\sum_{D=2,4,\ldots}\delta_{V,us,D}^{(1),kl}\right). \tag{15}
$$

and

$$R^{(0)kl}_{S,V} = |V_{us}|^2 \left(\sum_{D=2,4,\dots} \delta^{(0),kl}_{V,us,D} \right). \tag{16}$$

Thus, in our notation we have the relation

$$\delta^{kl}_{V,us,2} = a_{kl}\delta^{(1),kl}_{V,us,2} + \delta^{(0),kl}_{V,us,2}. \tag{17}$$

3 Mass Terms in Perturbative QCD

We would like to stress that inclusion of Cabibbo suppressed modes into the analysis of observables related to τ lepton decays gives not only an additional set of experimental data but open conceptually new possibilities because the massive piece can be measured in conjunction with the massless contribution thus providing a strict normalization and reducing the systematic errors of the experimental data. In this section we therefore compute mass corrections to the moments of order (m_s^2/M_τ^2) using first finite order perturbation theory and then the resumed perturbation theory with the evaluation of α_s and the quark mass treated exactly (neglecting only unknown higher order corrections to the β-function and the quark mass anomalous dimension).

3.1 Finite order analysis

Just like the perturbative predictions for the massless correlators also the quark mass corrections for the vector and axial correlators are identical[2] for the case under consideration with $m_i = m_s \neq 0$ and $m_j = m_u = m_d = 0$. The perturbative prediction for the quadratic mass corrections up to order α_s^3 and for arbitrary quark masses has been presented in [16] for the transversal piece of the correlator. The longitudinal piece is related to the scalar correlator through the Ward identity (5), viz.

$$q^4\Pi^{[2]}_{us,V/A} + q^2\Pi^{[1]}_{us,V/A} = q^4\Pi^{(0)}_{V/A} = m_s^2\Pi^{\mathrm{S/P}} + m_s(\langle\overline{s}s\rangle \mp \langle\overline{u}u\rangle). \tag{18}$$

The vacuum expectation values on the r.h.s. can be understood within the framework of perturbation theory and minimal subtraction. Then formally the last term in (18) is of order m^4. Working only within the second order in quark masses, the vacuum expectation values on the right hand side of (18) can be safely discarded at this point. We return to them in the section for nonperturbative contributions. Thus, the $\mathcal{O}(m_s^2)$ contribution to the longitudinal structure function $\Pi^{(0)}$ can be taken from Ref. [17] where the massless scalar correlator has been computed at α_s^3.

[2]This is strictly true only for the perturbative contributions.

The resulting polarization functions $\Pi^{[l]}_{V/A}$, $l=1,2$ is conveniently represented in the form

$$(Q^2)^{(l-2)}\Pi^{[l]}_{us,V/A}(q) = \frac{3}{16\pi^2}\Pi^{[l]}_{V/A,0}(\frac{\mu^2}{Q^2},\alpha_s) + \frac{3}{16\pi^2}\sum_{D\geq 2} Q^{-D}\Pi^{[l]}_{V/A,D}(\frac{\mu^2}{Q^2},m_s^2,\alpha_s). \tag{19}$$

Here the first term on the rhs corresponds to the massless limit while the first term in the sum stands for quadratic mass corrections. A similar decomposition is assumed for the polarization functions $\Pi^{(S/P)}_{us}(q)$. The results for both polarization functions with $D=2$ read

$$\begin{aligned}
\Pi^{[1]}_{V,2} = 2m_s^2 \Bigg\{ & l_{\mu Q} + \frac{\alpha_s}{\pi}\left[\frac{25}{4} - 4\,\zeta(3) + \frac{5}{3}l_{\mu Q} + l^2_{\mu Q}\right] \\
+ & \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{18841}{432} - \frac{1}{360}\pi^4 - \frac{3607}{54}\,\zeta(3) + \frac{1265}{27}\,\zeta(5)\right. \\
& \qquad \left. + \frac{4591}{144}l_{\mu Q} - \frac{35}{2}\,\zeta(3)l_{\mu Q} + \frac{22}{3}l^2_{\mu Q} + \frac{17}{12}l^3_{\mu Q}\right] \\
+ & \left(\frac{\alpha_s}{\pi}\right)^3 \left[l_{\mu Q}\left(\frac{1967833}{5184} - \frac{1}{36}\pi^4 - \frac{11795}{24}\,\zeta(3) + \frac{33475}{108}\,\zeta(5)\right.\right. \\
& \qquad \left.\left. + \frac{4633}{36}l_{\mu Q} - \frac{475}{8}\,\zeta(3)l_{\mu Q} + \frac{79}{4}l^2_{\mu Q} + \frac{221}{96}l^3_{\mu Q}\right) + k_3^{[1]}\right]\Bigg\},
\end{aligned} \tag{20}$$

$$\begin{aligned}
\Pi^{[2]}_{V,2} = -4m_s^2 \Bigg\{ & 1 + \frac{\alpha_s}{\pi}\left[\frac{7}{3} + 2l_{\mu Q}\right] \\
+ & \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{13981}{432} + \frac{323}{54}\,\zeta(3) - \frac{520}{27}\,\zeta(5) + \frac{35}{2}l_{\mu Q} + \frac{17}{4}l^2_{\mu Q}\right] \\
+ & \left(\frac{\alpha_s}{\pi}\right)^3 \left[l_{\mu Q}\left(\frac{14485}{54} + \frac{3659}{108}\,\zeta(3) - \frac{3380}{27}\,\zeta(5) + \frac{1643}{24}l_{\mu Q}\right.\right. \\
& \qquad \left.\left. + \frac{221}{24}l^2_{\mu Q}\right) + k_3^{[2]}\right]\Bigg\}.
\end{aligned} \tag{21}$$

Here $l_{\mu Q} = \ln\frac{\mu^2}{Q^s}$, the mass m_s as well as QCD coupling constant α_s are understood to be taken at a generic value of the t' Hooft mass μ. All correlators are renormalized within $\overline{\text{MS}}$-scheme. Note that terms of order α_s^3 are known only with "logarithmic" accuracy, that is the constant parts $k_3^{[1]}$ and $k_3^{[2]}$ in the large Q behavior of the corresponding correlators are not available[3].

[3]In fact the very calculation of the constant parts is well beyond the present calculational techniques.

This means that we do not know the $O(m_s^2\alpha_s^3)$ contributions to R_τ; however these constants (as well as the also unknown "low-energy" constant $\Pi^{[1]}(0)$) do not appear in the moments $R_\tau^{(1,0)k,l}$ with $k \geq 0, \quad l \geq 1$.

Let us discuss some concrete results (neglecting for the moment any nonperturbative condensate contributions and quartic mass corrections as well). Since the perturbative result, in particular the m^2 terms, do not differentiate between vector and axial vector channels we will consider in the following their sum. The mass corrections to the moments of spin 0 and spin 1 final state distributions can now be cast into the following form ($m_s = m_s(M_\tau)$, $\alpha_s = \alpha_s(M_\tau)$)

$$\delta_{us,2}^{kl} = b_0 \frac{m_s^2}{M_\tau^2}\{1 + b_1 \frac{\alpha_s}{\pi} + b_2 \left(\frac{\alpha_s}{\pi}\right)^2 + b_3 \left(\frac{\alpha_s}{\pi}\right)^3\}. \tag{22}$$

In view of the large size of the coefficients b_i we attempt to reduce the higher order corrections by adopting the value 1 GeV for the renormalization scale of the running mass and define an alternative set of coefficients $\hat{b}_i$ through

$$\delta_{us}^{kl} = \hat{b}_0 \frac{m_s^2(1\ \ \mathrm{GeV})}{M_\tau^2}\{1 + \hat{b}_1 \frac{\alpha_s}{\pi} + \hat{b}_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \hat{b}_3 \left(\frac{\alpha_s}{\pi}\right)^3\}. \tag{23}$$

The results are listed in Table 1. The same expansions can also be obtained for the correction to the spin zero and spin one parts separately. In view of the additional nonperturbative contribution $\sim \Pi^{[1]}(0)$ which appears in the lowest order moment ($l = 0$) due to the spin separation only the results for $l \geq 1$ are given in Tables 2-3 for these latter cases. The additional nonperturbative contribution $\sim \Pi^{[1]}(0)$ in the axial vector current can be estimated in the chiral limit and happens to be connected to the contribution of the pseudoscalar meson (pion or kaon) to the correlator. The distinguished role of the pion is due to its nature as a Goldstone particle related to spontaneous violation of the chiral symmetry of QCD. When the explicit violation of the chiral symmetry given by nonzero values of quark masses is small these masses provide small corrections to the massless limit (pion dominance) that can be accounted for on a regular basis within Chiral Perturbation Theory [18]. As for corresponding quantity in the vector channel there is no solid physical arguments for estimating its value with finite masses of quarks though it vanishes in the massless limit due to vector current conservation and the pattern of spontaneous symmetry breaking.

3.2 Resummation of effects of running

The rapid change of coefficients of perturbation theory expansions for different moments is caused by the running of the coupling constant and the mass along the contour of integration. The resummation of these effects can be performed in all orders of α_s. The technique for the massless case is described in the literature

[12, 15, 19] so here we concentrate on the massive case. It introduces no much new technically but extends the freedom of choosing the resummation procedure due to additional parameter – the running mass of a quark.

Let us, nevertheless, recall the central idea for the massless case. The main object to start with is the Adler's D-function for the transverse part of the correlator. The orders of perturbation theory in the $\overline{\rm MS}$ scheme are formally counted after the renormalization group improvement of the series. The information from the perturbative treatment is contained in the string of the coefficients of powers of the running coupling. The polarization function that appears in the integral over the large circle is then restored from the D-function (including renormalization group improvement) by solving the corresponding evolution equation exactly with the β function taken to a given fixed order in α_s. Let us demonstrate the procedure in a simple example. Consider a D-function with its leading term normalized to unity (here and below we are using $a \equiv \alpha_s/\pi$)

$$D(Q^2) = 1 + a(Q^2) + k_1 a(Q^2)^2 + k_2 a(Q^2)^3 + k_3 a(Q^2)^4 + \dots . \tag{24}$$

It is connected to $\Pi(Q^2)$ by

$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2). \tag{25}$$

In leading order for the β function

$$\beta(a) = -\beta_0 a^2, \quad Q^2 \frac{d}{dQ^2} a(Q^2) = \beta(a(Q^2)) \tag{26}$$

the polarization function can be completely restored in the closed form. Direct integration of Eq. (25) gives the improved polarization function

$$\begin{aligned} \Pi(Q^2) &= \ln(\mu^2/Q^2) + \Pi(\mu^2) \\ &+ \frac{1}{\beta_0}\left(-\ln \frac{1}{a(Q^2)} + k_1 a(Q^2) + \frac{k_2}{2} a(Q^2)^2 + \frac{k_3}{3} a(Q^2)^3 + \dots \right) \end{aligned} \tag{27}$$

that ought to be integrated along the contour in the complex plane with

$$a(Q^2) = \frac{a(\mu^2)}{1 + \beta_0 a(\mu^2) \ln(Q^2/\mu^2)} \tag{28}$$

which can be continued into the complex Q^2 plane. The normalization point μ is conveniently chosen to be equal to the τ lepton mass $\mu = M_\tau$. The part independent of Q^2 (the integration constant $\Pi(\mu^2)$) does not contribute to the integral. This is also obvious from the spectral integral itself because $\Pi(\mu^2)$ being a constant has no discontinuity across the physical cut in the Q^2 complex plane. The first term in (27) is simply the partonic contribution $\ln(\mu^2/Q^2)$.

The generalization to higher orders of β function is straightforward. In second order the integration of Eq. (25) can be still performed explicitly while for third and fourth order of the β function numerical integration is more convenient.

To make things clearer let us show the explicit example in the second order for the β function

$$\beta(a) = -\beta_0 a^2 - \beta_1 a^3. \tag{29}$$

(In higher orders the β-function is scheme dependent. However, there is always a scheme where this is the complete expression for the β function – the t' Hooft scheme. Then the convention could be to resum always in this scheme that is justified by technical simplicity. Nevertheless, we however stick to the $\overline{\mathrm{MS}}$- scheme in the present paper.) A contribution to the D-function of the form $\Delta D(Q^2) = a(Q^2)^2$ leads after integration of Eq. (25) to the corresponding contribution to the polarization function $\Delta\Pi(Q^2)$ of the form

$$\Delta\Pi(Q^2) = \frac{1}{\beta_1}\ln\left(a(Q^2) + \frac{\beta_0}{\beta_1}\right) + \Delta\Pi(\mu^2) \tag{30}$$

as can be easily checked by direct differentiation. The improved polarization function $\Delta\Pi(Q^2)$ (30) can then be used for integration along the contour. The remark about the constant terms applies again.

The appearance of mass introduces another freedom in the choice of the basic quantities that accumulate the perturbative information. The actual procedure is described below. As basic quantities we choose $\Pi^{[1]}_{us}(q^2)$ and $\Pi^{[2]}_{us}(q^2)$. The renormalization of the pieces proportional to m_s^2 is different for $\Pi^{[1]}$ and $\Pi^{[2]}$. The second one $\Pi^{[2]}_{us,2}(Q^2)$, is scale-invariant and the renormalization group improvement can be performed directly

$$Q^2\Pi^{[2]}_{us,2}(Q^2) = k_0^{[2]} m_s^2(Q^2)(1 + k_1^{[2]} a(Q^2) + \ldots). \tag{31}$$

In contrast $\Pi^{[1]}$ is not renormalized multiplicatively and the corresponding renormalization group equation is not uniform i.e it has a free term. The problem is solved by introducing the corresponding D-function $D^{[1]}(Q^2)$ by one differentiation with respect to Q^2. The result is

$$D^{[1]}_{us,2}(Q^2) = -\frac{Q^2}{2}\frac{d}{dQ^2}\Pi^{[1]}_{us,2}(Q^2) = m_s^2(Q^2)(1 + d_1^{[1]} a(Q^2) + \ldots). \tag{32}$$

Then we proceed as explained before in construction of the polarization operator. The running of the mass as taken into account through the renormalization group equation

$$Q^2\frac{d}{dQ^2} m_s(Q^2) = \gamma_m(a(Q^2)) m_s(Q^2), \quad \gamma_m(a) = -\gamma_0 a - \gamma_1 a^2 - \ldots \tag{33}$$

with the solution

$$m_s(Q^2) = m_s(\mu^2) \exp \int_{\alpha_s(\mu^2)}^{\alpha_s(Q^2)} \frac{\gamma(x)dx}{\beta(x)}. \tag{34}$$

Subsequently, the integration along the contour can be performed directly.

The explicit formula in the leading order of the β function for $\Pi^{[2]}(Q^2)$ is easily found. Having solved the Eq. (33) for the leading order β and γ functions (or using the explicit formula (34)) we have

$$Q^2 \Pi^{[2]}_{us,2}(Q^2) = k_0^{[2]} m_s^2(\mu^2) \left(\frac{a(Q^2)}{a(\mu^2)}\right)^{\frac{2\gamma_0}{\beta_0}} (1 + k_1^{[2]} a(Q^2) + \ldots) \tag{35}$$

where $a(Q^2)$ is the solution to the renormalization group Eq. (28).

For the second amplitude the result is slightly more complicated, namely

$$\Pi^{[1]}_{us,2}(Q^2)$$

$$= -\frac{m_s^2(\mu^2)}{\beta_0} \left(\frac{a(Q^2)}{a(\mu^2)}\right)^{\frac{2\gamma_0}{\beta_0}} \left(\frac{1}{\left(\frac{2\gamma_0}{\beta_0} - 1\right)} \frac{1}{a(Q^2)} + \frac{d_1^{[1]}}{\frac{2\gamma_0}{\beta_0}} + \frac{d_2^{[1]}}{\frac{2\gamma_0}{\beta_0} + 1} a(Q^2) + \ldots \right). \tag{36}$$

These formulae should be substituted in (10),(11) and integrated along the contour. The generalization to higher orders of the β and γ functions is straightforward. Again up to the second order expansion of these functions in α_s explicit analytical integration can be performed in terms of elementary functions for some amplitudes. (Third order also allows some integrations in terms of elementary functions but formulas become too awkward and the numerical treatment is preferable.) The coefficients $d_i^{[1]}$ can be inferred from the explicit expression for the polarization function (20). They accumulate the whole perturbative information because the β-function and the quark anomalous dimension necessary for the restoration of the full expressions (20) and (21) are known.

We perform the analysis along these lines for the same moments as in the finite order of perturbation theory with the available coefficients for $\Pi^{[2]}$ and for the D functions in the case of $\Pi^{[1]}$ and with β and γ functions from first to fourth order [20, 21, 22].

For the presentation it is convenient to order the results according to the corresponding contributions from D/Π functions and to attach a power of $\alpha_s(M_\tau^2)$ to every consequent term as a mark.

For instance the typical answer analogous to Eqs. (22),(23) can then be cast into the form (here and below $m_s = m_s(M_\tau)$)

$$\tilde{\delta}^{kl}_{us} = \tilde{b}_0(\alpha_s) \frac{m_s^2}{M_\tau^2} \{1 + \tilde{b}_1(\alpha_s) \frac{\alpha_s}{\pi} + \tilde{b}_2(\alpha_s) \left(\frac{\alpha_s}{\pi}\right)^2 + \tilde{b}_3(\alpha_s) \left(\frac{\alpha_s}{\pi}\right)^3 \} \tag{37}$$

where the coefficients $\tilde{b}_i(\alpha_s)$ are obtained with resummation and become functions of α_s. These functions are not given in an explicit form but are fixed numbers obtained from the actual analysis because the contour integration was not done in general analytic form but only numerically. The resummation procedure is expected to exhibit an improved behaviour of the series compared to the finite order analysis and to provide a definite coefficient in front of the free parameter m_s^2.

The results are presented in Tables 3-6. By comparing the results of these two analyses one can judge the degree of improvement achieved by resummation in $\overline{\rm MS}$ scheme.

4 Non-perturbative corrections for $J = 0$ moments

Besides the perturbative radiative corrections to R_τ also nonperturbative QCD effects influence the hadronic τ decay rate and its differential distributions analyzed with the moments. The short distance operator product expansion (OPE) for the spectral functions

$$\Pi^{(J)}(s=-Q^2) = \sum_{D=0,2,4,\cdots} \frac{1}{(Q^2)^{D/2}} \sum_{\dim\mathcal{O}=D} C^{(J)}(Q^2,\mu)\langle\mathcal{O}(\mu)\rangle \tag{38}$$

may be used to take into account both perturbative and nonperturbative contributions. Here we collect the known non-perturbative corrections to the structure function $\Pi^{(0)}_{V/A}$.

4.1 Dimension $D = 4$ and $D = 6$ Corrections

Due to Ward identity (18) the $D = 4$ part of the polarization function $\Pi^{(0)}_{V/A}$ can be cast into the following convenient form

$$\Pi^{(0)}_{V/A,4}(Q^2) = \frac{m_s^2\Pi_{(S/P),2}(Q^2)}{Q^4} + \frac{m_s(\langle\bar{s}s\rangle \mp \langle\overline{u}u\rangle)}{Q^4}\frac{16\pi^2}{3} \tag{39}$$

where $\Pi_{(S/P),2}$ is [25]

$$\begin{aligned}
\Pi_{S/P,2} = m_s^2 \Big\{ & -4 - 4l_{\mu Q} \\
+ & \frac{\alpha_s}{\pi}\left[-\frac{100}{3} + 16\,\zeta(3) - \frac{64}{3}l_{\mu Q} - 8l_{\mu Q}^2\right] \\
+ & \left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{33109}{108} + \frac{2}{135}\pi^4 + \frac{1978}{9}\zeta(3) - \frac{620}{9}\zeta(5)\right.
\end{aligned}$$

$$
- \frac{5065}{18} l_{\mu Q} + \frac{308}{3}\zeta(3) l_{\mu Q} - 97 l_{\mu Q}^2 - \frac{50}{3} l_{\mu Q}^3 \Big] \Big\} , \qquad (40)
$$

$$
= -4\left(1. + 3.5251\frac{\alpha_s}{\pi} + 28.0923\left(\frac{\alpha_s}{\pi}\right)^2\right). \qquad (41)
$$

In the last line we have set $\mu^2 = Q^2$. Note that there are no corrections to the coefficient function of the quark condensate term in Eq. (39) because it comes from equal-time commutation relation for the quark fields that are just initial conditions for the field theory.

It is clear from (18) that these power corrections are essentially given by the corresponding $D = 4$ terms of the OPE for the structure function $\Pi^{(S)}_{V/A}$ which are well-known (see, e.g. Refs. [23, 25] and references therein). As a result we have

$$
\begin{aligned}
\frac{3}{16\pi^2}\Pi^{(0)}_{V/A,6} = m_s^2 \Big[& + \frac{1}{8}\left(1 + \frac{\alpha_s}{\pi}\frac{11}{2} + 2\frac{\alpha_s}{\pi} l_{\mu Q}\right)\frac{\alpha_s}{\pi}\langle GG\rangle \\
& \pm \left(1 + \frac{14}{3}\frac{\alpha_s}{\pi} + 2\frac{\alpha_s}{\pi} l_{\mu Q}\right) m_s\langle \bar u u\rangle \\
& + \frac{1}{2}\left(1 + \frac{11}{3}\frac{\alpha_s}{\pi} + 2\frac{\alpha_s}{\pi} l_{\mu Q}\right) m_s\langle \bar s s\rangle \\
& + \frac{3}{16\pi^2} m_s^4 \left(1 - 2 l_{\mu Q} + \frac{\alpha_s}{\pi}\left[8\zeta(3) - 6 - 4 l_{\mu Q} - 8 l_{\mu Q}^2\right]\right)\Big].
\end{aligned} \qquad (42)
$$

The corresponding contributions to the $(k, l) = (0, 1)$ moment read (to save space we have partially converted the coefficient in front of m_s^4 term to numbers and put $\mu = M_\tau$ so that below $\alpha_s = \alpha_s(M_\tau)$ and $m_s = m_s(M_\tau)$).

$$
\delta^{(0),01}_{us,4} = \frac{45}{2}\frac{m_s^4}{M_\tau^4}\left[1. + 4.77815\frac{\alpha_s}{\pi} + 31.0478\left(\frac{\alpha_s}{\pi}\right)^2\right] - 12\pi^2\frac{(\langle m_s \bar s s\rangle \mp \langle m_s \bar u u\rangle)}{m_s^4} \qquad (43)
$$

and

$$
\begin{aligned}
\delta^{(0),01}_{us,6} = \frac{m_s^2}{M_\tau^6}\Big[& -3 \quad \pi^2\left(1 + \frac{\alpha_s}{\pi}\frac{11}{2}\right)\langle\frac{\alpha_s}{\pi} GG\rangle \\
& \mp \quad 24\pi^2\left(1 + \frac{14}{3}\frac{\alpha_s}{\pi}\right)\langle m_s \bar u u\rangle \\
& - \quad 12\pi^2\left(1 + \frac{11}{3}\frac{\alpha_s}{\pi}\right)\langle m_s \bar s s\rangle \\
& - \quad \frac{9}{2} m_s^4\left(1 + \frac{\alpha_s}{\pi}\left[8\zeta(3) + \frac{8}{3}\pi^2 - 22\right]\right)\Big].
\end{aligned} \qquad (44)
$$

Note, please, that in Eqs. (43,44) we have used the very quark and gluon condensates normalized at the "natural" scale $\hat\mu = M_\tau$ for our RG invariant condensates (for more details, see [9, 11, 24]):

$$
\begin{aligned}
\langle\frac{\alpha_s}{\pi}GG\rangle = &\quad \langle\frac{\alpha_s(M_\tau)}{\pi}GG(\mu = M_\tau)\rangle, \\
\langle m_i\bar{\Psi}_j\Psi_j\rangle = &\quad \langle m_i(M_\tau)\bar{\Psi}_j\Psi_j(\mu = M_\tau)\rangle.
\end{aligned} \tag{45}
$$

5 Discussion of numerical results

Let us first consider the mass corrections to the total rate within fixed order perturbation theory. The coefficients b_i are already fairly large for all the moments. Even worse, however, is the fact that the series grows dramatically for all the moments, rendering a mass determination from the total rate or the corresponding momenta potentially unreliable. The transition from $m_s(M_\tau)$ to $m_s(1\text{ GeV})$ reduces the coefficients only marginally. From Table 2 it is evident that the large corrections originate mainly from the spin zero contribution to the moments, leaving significantly smaller QCD corrections to the spin one part. Hence, it is in principle possible to determine m_s from the spin one part alone. Alternatively one might try to combine spin zero and spin one moments with different relative coefficients to artificially decrease the QCD corrections further. However, in the absence of any physically motivated reasonings we refrain from such an approach. The main problem of the strange quark determination and even the total perturbative analysis of the Cabibbo suppressed τ lepton decays is the interpretation of the perturbation theory series for m_s^2 corrections. In passing we note that also perturbative corrections to the power suppressed terms proportional to m_s^4 and m_s^6 (see Eqs. (43,44)) might provide us with another example of bad behaviour of higher order terms. However, since the $\mathcal{O}(\alpha_s^2$ corrections to the spin one (and, as a consequence, to the total R_τ) are available we concentrate below at discussing the m_s^2 terms only.

The problem is fairly obvious from a consideration of the two lowest order moments. Let us use $\alpha_s(M_\tau) = 0.334$ [26] in computing the size of the perturbative contributions. The order of magnitude of the unknown coefficient $k_3^{[2]}$ in Eq. (21) is estimated on the basis of a geometric series to amount to $k_3^{[2]} = 0{\pm}19.6^2/2.33 = 0 \pm (k_2^{[2]})^2/k_1^{[2]} = 0 \pm 164.4$. For fixed order we thus find series is thus given by (below $a = \alpha_s(M_\tau)/\pi$ and $m_s = m_s(M_\tau)$)

$$
\begin{aligned}
\delta^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1. + 5.33a + 46.0a^2 + 284a^3 + 0.75a^3k_3^{[2]}) \\
&= -8\frac{m_s^2}{M_\tau^2}(1. + 0.567 + 0.520 + 0.341 \pm 0.148) \\
&= -8\frac{m_s^2}{M_\tau^2}(2.4 \pm 0.5),
\end{aligned} \tag{46}
$$

where we have assumed the (maximal!) value of the $\mathcal{O}(\alpha_s^3)$ term as an estimate of the theoretical uncertainty (this convention will be used also below). For the "contour improved" series one obtains

$$\begin{aligned}
\tilde{\delta}^{00}_{us,2} &= -8\frac{m_s^2}{M_\tau^2}(1.44 + 3.65a + 30.9a^2 + 72.2a^3 + 1.18a^3 k_3^{[2]}) \\
&= -8\frac{m_s^2}{M_\tau^2}(1.44 + 0.389 + 0.349 + 0.0867 \pm 0.234) \\
&= -8\frac{m_s^2}{M_\tau^2}(2.26 \pm 0.32).
\end{aligned} \tag{47}$$

Comparing the "improved" with the finite order analysis one observes that higher orders give numerically smaller contributions although the apparent convergence is rather marginal as well. The ratio between two correction terms $\tilde{\delta}^{00}_{us,2}/\delta^{00}_{us,2} \approx 0.95$ shows a relatively stable behaviour. The results within resummation technique are stable within the allowed range of $\alpha_s = 0.312 - 0.356$ as checked by direct computation. Although no firm prediction can be made as a consequence of the large corrections, it is at least encouraging to observe that the sign of the mass correction remains preserved and the consecutive terms exhibit a marginal decrease. Eq. (47), with the uncertainty increased by perhaps a factor 2 could be considered as a reasonable estimate of the strange mass corrections.

In fixed order approximation the moments with $l \geq 1$ are independent of the constant $k_3^{[2]}$; a residual dependence remains, however, in the "contour improved" treatment

$$\begin{aligned}
\delta^{01}_{us,2} &= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(1 - 4.17a - 113.a^2 - 1820.a^3) \\
&= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(1 - 0.443 - 1.27 - 2.19) \\
&= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(-2.9 \pm 2.2),
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\tilde{\delta}^{01}_{us,2} &= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(-2.26 - 14.7a - 204.a^2 - 171.a^3 - 13.3a^3 k_3^{[2]}) \\
&= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(-2.26 - 1.56 - 2.3 - 0.206 \mp 2.62) \\
&= -\frac{5}{3}\frac{m_s^2}{M_\tau^2}(-6.3 \pm 2.8).
\end{aligned} \tag{49}$$

The $(0,1)$ moments thus exhibit a rapid growth of the coefficients and, at the same time, with $\tilde{\delta}^{01}_{us,2}/\delta^{01}_{us,2} = 2.2$ a strong dependence on the improvement procedure. This comparison shows that there is no consistent prediction in $\overline{\mathrm{MS}}$ scheme for this observable – the first moment of the differential rate.

Now we turn to to the contributions of the spin one and spin zero separately. As stated before, the lowest moments ($l = 0$) of the spin-dependent functions depend on the nonperturbative quantity $\Pi^{[1]}(0)$. For the spin one part and for $(k, l) = (0, 1)$ we find

$$\begin{aligned}
\delta^{(1)01}_{us,2} &= -5\frac{m_s^2}{M_\tau^2}(1. + 4.83a + 35.7a^2 + 276.a^3) \\
&= -5\frac{m_s^2}{M_\tau^2}(1. + 0.514 + 0.404 + 0.331) \\
&= -5\frac{m_s^2}{M_\tau^2}(2.25 \pm 0.33) \qquad (50)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\delta}^{(1)01}_{us,2} &= -5\frac{m_s^2}{M_\tau^2}(1.37 + 2.55a + 16.1a^2 + 135a^3) \\
&= -5\frac{m_s^2}{M_\tau^2}(1.37 + 0.271 + 0.182 + 0.163) \\
&= -5\frac{m_s^2}{M_\tau^2}(2.0 \pm 0.2). \qquad (51)
\end{aligned}$$

Note that spin 1 contribution is determined by the component $\Pi^{[1]}$ alone and is known up to third order. Clearly, this series is decreasing in a reasonable way (comparable to the behaviour of $\tilde{\delta}^{00}_{us,2}$) and, at the same time, only moderately dependent on the improvement prescription with $\tilde{\delta}^{(1)01}_{us,2}/\delta^{(1)01}_{us,2} = 0.89$. On the basis of Eq. (51) this moment might well serve for a reliable m_s determination, with a sufficiently careful interpretation of the theoretical uncertainty.

The corresponding spin zero part is, per se, proportional to m_s^2 (not counting the "condensate" contributions discussed in Section 4) and thus could be considered as ideal for a measurement of m_s. However, the behaviour of the perturbative series

$$\begin{aligned}
\delta^{(0)01}_{us,2} &= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(1. + 9.33a + 110a^2 + 1323a^3) \\
&= \frac{m_s^2}{M_\tau^2}(1. + 0.992 + 1.24 + 1.59) \\
&= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(4.8 \pm 1.6) \qquad (52)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\delta}^{(0)01}_{us,2} &= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 11.2a + 126.a^2 + 289.a^3 + 6.63a^3k_3^{[2]}) \\
&= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(3.19 + 1.19 + 1.42 + 0.347 \pm 1.31)
\end{aligned}$$

$$= \frac{3}{2}\frac{m_s^2}{M_\tau^2}(6.14 \pm 1.6) \tag{53}$$

shows a rapid growth of the coefficients. The series is not expected to provide an accurate prediction for the mass effects. (The same rapid growth of the coefficients in the perturbative series is present for the scalar correlator related to $\Gamma(H \to b\bar{b})$ [17].) Nevertheless, an interesting estimate could also be deduced from the spin zero contributions, in particular after resummation. For the spin zero, spin one and the total rate the following relation holds

$$\delta^{01}_{us,2} = \delta^{(1)01}_{us,2} + \frac{20}{9}\delta^{(0)01}_{us,2}\,.$$

Besides the observables themselves one has also look at convergence of the β and γ functions that determine the running along the contour. In the present case

$$\beta(a) = -2.25a^2(1 + 1.78a + 4.47a^2 + 21.0a^3)$$

or at $a = 0.1$

$$\beta(0.1) = -0.0225(1 + 0.18 + 0.045 + 0.021)$$

which is quite good. For the γ function, however,

$$\gamma(a) = -a(1 + 3.79a + 12.4a^2 + 44.3a^3)$$

and the convergence is marginally acceptable.

The resumed series behave in general better than those of finite order. However, for the mass corrections they still do not satisfy the heuristic criteria of convergence. In practice the resummation maintains the convergence pattern of the corresponding D-function. For the mass corrections the D-functions themselves exhibit rapidly growing coefficients of the perturbative series, whence the resummation does not lead to a significant improvement. Numerically the convergence for D-functions in $\overline{\rm MS}$ scheme is marginal. For the present case the D-function for $\Pi^{[1]}(Q^2)$ is given by

$$\begin{aligned} D^{[1]}(Q^2) = m_s^2(Q^2)(1 &+ \frac{5}{3}a + a^2\left(\frac{4591}{144} - \frac{35}{2}\zeta(3)\right) \\ &+ a^3\left(\frac{1967833}{5184} - \frac{\pi^4}{36} - \frac{11795}{24}\zeta(3) + \frac{33475}{108}\zeta(5)\right) \\ &= m_s^2(Q^2)(1 + 1.67a + 10.84a^2 + 107.53a^3) \end{aligned}$$

whereas a much better pattern of convergence is observed for massless part

$$\begin{aligned} D_0(Q^2) = 1 + a + a^2\left(\frac{299}{24} - 9\zeta(3)\right) + a^3\left(\frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5)\right) \\ = 1 + a + 1.64a^2 + 6.37a^3\,. \end{aligned}$$

This is the reason why the precise determination of the strong coupling constant from the τ lepton lifetime is possible.

6 Summary

Tau decays in the $\Delta S = 1$ channel are sensitive to the mass of the strange quark. Moments of the four spectral functions corresponding to spin zero and spin one contributions, induced by vector and axial vector currents can be evaluated in perturbative QCD. With this motivation we have evaluated the QCD corrections up to order α_s^2 for the rate and up to order α_s^3 for the higher moments of the spectral function, separated according to the spin zero and spin one contributions. Resummation of the RG enhanced terms has been applied to the various monets.

At first glance, the spin zero part of the rate and of the moments appears to be particularly promising for an m_s determination, since this piece is multiplied by m_s^2 in the parton model. However, as a consequence of the spin zero separation only moments with $l > 0$ can be calculated in pQCD. In addition, it turns out that the perturbative corrections to the spin zero part are remarkably large, invalidating the straightforward extraction of m_s from such an analysis.

Also the total rate, i.e. the combination of spin zero and spin one contributions, exhibits a small mass dependent term which can, in principle, also be used to determine m_s. However, again, one observes remarkably large QCD corrections which can be traced—at least for the higher moments—to the spin zero part. This is clearly visible by considering the mass terms of the separated spin one piece, which exhibit reasonably smaller coefficients of the first, second and the third order corrections.

Resummation leads to a modest improvement of the apparent convergence of the series, such that the lowest moments can be predicted with acceptable accuracy. Assuming sufficiently precise data, a combined analysis of the spin zero and one moments might thus lead to a reliable determination of the strange quark mass.

We do not discuss μ (renormalization scale) dependence in $\overline{\text{MS}}$ scheme as a special topic fixing it always at $\mu = M_\tau$. Our estimate of the error for "contour improved" analysis (2.26 ± 0.6) is rather conservative and covers all reasonable change of μ. The more general topic of investigating the entire scheme dependence of this quantity is outside the scope of the paper.

To conclude we note that the large value for the correction factor $\tilde{\delta}^{00}_{us,2}/(-8)$ leads us to a reduction of the m_s-value as determined from the data [8] on the basis of the earlier calculation [9]. The reduction by about 15% leads to $m_s(M_\tau) = (150 \pm 30_{exp} \pm 20_{th})$ MeV. In view of the large corrections the theoretical uncertainity can just be considered as a guess based on Eq. (47) and the subsequent discussion. This corresponds $m_s(1 \text{ GeV}) = 200 \pm 40_{exp} \pm 30_{th}$ MeV in good agreement with other determinations (see, e.g. [27, 23, 24]).

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Note added. Just before completing this paper we became aware of a paper [28] by Pich and Prades where mass corrections to the total τ decay rate have been discussed.

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Table 1: m_s^2 corrections to R_τ and its moments; for the entries with $l = 0$ the unknown coefficinet $k_3^{[2]}$ is set to zero

k	l	b_0	b_1	b_2	b_3	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
0	0	-8	5.333	46.	283.6	3.034	24.68	92.82
0	1	$-\frac{5}{3}$	-4.167	-112.8	$-1818.$	-6.466	-112.3	$-1558.$
0	2	$-\frac{3}{2}$	-1.767	-54.9	-711.4	-4.066	-59.89	-605.8
0	3	$-\frac{7}{5}$	-1.1	-40.24	-474.3	-3.4	-46.76	-408.5
1	0	$-\frac{75}{7}$	5.967	56.59	423.7	3.667	33.81	202.9
1	1	$-\frac{9}{5}$	-5.767	-151.4	$-2556.$	-8.066	-147.2	$-2193.$
1	2	$-\frac{21}{13}$	-2.433	-69.57	-948.5	-4.733	-73.02	-803.2
1	3	$-\frac{3}{2}$	-1.481	-48.58	-601.7	-3.781	-54.22	-513.2
2	0	$-\frac{27}{2}$	6.456	65.25	547.8	4.156	41.36	302.7
2	1	$-\frac{21}{11}$	-7.433	-192.3	$-3360.$	-9.733	-184.3	$-2887.$
2	2	$-\frac{12}{7}$	-3.148	-85.31	$-1209.$	-5.447	-87.12	$-1021.$
2	3	$-\frac{27}{17}$	-1.898	-57.57	-740.8	-4.197	-62.26	-627.9
3	0	$-\frac{49}{3}$	6.852	72.61	659.5	4.553	47.8	393.8
3	1	-2	-9.148	-235.1	$-4221.$	-11.45	-223.1	$-3634.$
3	2	$-\frac{9}{5}$	-3.898	-101.9	$-1489.$	-6.197	$-102.$	$-1256.$
3	3	$-\frac{5}{3}$	-2.342	-67.12	-890.7	-4.642	-70.78	-751.9

Table 2: m_s^2 corrections to the spin zero part of R_τ

k	l	b_0	b_1	b_2	b_3	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
0	1	$\frac{3}{2}$	9.333	110.	1323.	7.034	79.47	948.4
0	2	$\frac{3}{8}$	7.833	72.96	622.1	5.534	45.89	346.7
0	3	$\frac{3}{20}$	7.233	60.19	427.2	4.934	34.5	186.6
1	1	$\frac{9}{8}$	9.833	122.3	1556.	7.534	90.67	1149.
1	2	$\frac{9}{40}$	8.233	81.47	752.	5.934	53.49	453.5
1	3	$\frac{3}{40}$	7.567	66.81	518.3	5.267	40.36	259.5
2	1	$\frac{9}{10}$	10.23	132.5	1757.	7.934	99.96	1323.
2	2	$\frac{3}{20}$	8.567	88.8	868.9	6.267	60.05	550.5
2	3	$\frac{3}{70}$	7.852	72.66	602.3	5.553	45.55	327.5
3	1	$\frac{3}{4}$	10.57	141.3	1935.	8.267	107.9	1477.
3	2	$\frac{3}{28}$	8.852	95.26	975.6	6.553	65.85	639.7
3	3	$\frac{3}{112}$	8.102	77.91	680.4	5.803	50.23	391.3

Table 3: m_s^2 corrections to the spin one part of R_τ

k	l	b_0	b_1	b_2	b_3	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
0	1	-5	4.833	35.73	275.6	2.534	15.56	113.
0	2	$-\frac{27}{8}$	3.567	16.13	29.42	1.267	-1.121	-76.63
0	3	$-\frac{14}{5}$	3.067	9.973	-23.6	0.767	-6.129	$-111.$
1	1	$-\frac{63}{10}$	5.376	44.13	381.1	3.076	22.71	194.3
1	2	$-\frac{105}{26}$	3.967	21.06	71.84	1.667	2.885	-49.16
1	3	$-\frac{13}{4}$	3.391	13.55	1.377	1.091	-3.295	-97.15
2	1	$-\frac{84}{11}$	5.817	51.34	477.7	3.517	28.91	270.3
2	2	$-\frac{33}{7}$	4.307	25.49	113.5	2.007	6.537	-20.81
2	3	$-\frac{63}{17}$	3.674	16.85	26.68	1.374	-0.6516	-81.99
3	1	-9	6.186	57.65	566.7	3.886	34.38	341.5
3	2	$-\frac{27}{5}$	4.602	29.53	154.	2.303	9.891	7.746
3	3	$-\frac{25}{6}$	3.925	19.9	51.96	1.625	1.825	$-66.$

Table 4: m_s^2 corrections to R_τ and its moments; $\alpha_s(M_\tau) = 0.334$; the unknown coefficient $k_3^{[2]}$ is set to zero.

(k,l)	$\tilde{b}_0$	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{\delta}_{us,2}/\delta_{us,2}$
$(0,0)$	-11.51	2.539	21.48	50.14	0.9325
$(0,1)$	3.768	6.499	90.06	75.69	2.18
$(0,2)$	-1.568	4.643	95.09	71.91	-4.184
$(0,3)$	-0.5285	-2.855	-80.94	-4.813	0.5966
$(1,0)$	-18.06	2.893	27.61	52.42	1.019
$(1,1)$	8.036	6.209	90.84	75.1	2.821
$(1,2)$	-2.768	6.295	133.9	88.81	-4.756
$(1,3)$	-0.1476	-33.51	-843.2	-289.1	2.851
$(2,0)$	-26.22	3.211	33.66	54.59	1.126
$(2,1)$	14.42	6.219	95.72	76.65	3.568
$(2,2)$	-5.014	7.299	158.5	98.35	-6.156
$(2,3)$	0.6619	18.48	459.2	197.5	4.709
$(3,0)$	-36.15	3.504	39.71	56.74	1.252
$(3,1)$	23.49	6.326	$102.$	78.82	4.453
$(3,2)$	-8.873	7.866	173.8	103.4	-8.211
$(3,3)$	2.204	11.94	299.5	137.6	7.139

Table 5: m_s^2 corrections to $R_\tau^{(0)}$ and its moments; $\alpha_s(M_\tau) = 0.334$, the unknown coefficient $k_3^{[2]}$ is set to zero.

(k,l)	$\tilde{b}_0$	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{\delta}_{us,2}/\delta_{us,2}$
$(0,1)$	4.779	3.506	39.55	90.6	1.274
$(0,2)$	0.2785	-3.148	-107.5	-230.4	-0.1803
$(0,3)$	0.2105	2.363	28.74	53.57	0.777
$(1,1)$	4.5	3.918	48.65	110.5	1.585
$(1,2)$	0.06802	-20.2	-529.1	$-1109.$	-0.6913
$(1,3)$	0.1234	3.124	50.53	92.52	1.041
$(2,1)$	4.432	4.288	57.51	129.2	1.955
$(2,2)$	-0.05534	31.79	763.	1569.	-1.388
$(2,3)$	0.08678	4.085	78.8	142.1	1.495
$(3,1)$	4.488	4.628	66.21	146.9	2.392
$(3,2)$	-0.1421	14.87	345.2	697.9	-2.318
$(3,3)$	0.06992	5.115	109.9	195.2	2.215

Table 6: m_s^2 corrections to $R_\tau^{(1)}$ and its moments; $\alpha_s(M_\tau) = 0.334$.

(k,l)	$\tilde{b}_0$	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{\delta}_{us,2}/\delta_{us,2}$
$(0,1)$	-6.852	1.861	11.77	98.8	0.8833
$(0,2)$	-2.961	0.9785	-0.1989	-70.28	0.5589
$(0,3)$	-2.493	1.257	5.49	41.2	0.7861
$(1,1)$	-9.965	2.071	14.62	139.	0.9712
$(1,2)$	-3.5	0.7497	-4.874	-161.9	0.412
$(1,3)$	-3.026	1.337	6.927	73.89	0.8044
$(2,1)$	-13.78	2.269	17.54	184.1	1.081
$(2,2)$	-3.907	0.36	-12.71	-318.4	0.2254
$(2,3)$	-3.626	1.458	9.37	131.9	0.861
$(3,1)$	-18.39	2.458	20.54	233.9	1.213
$(3,2)$	-4.098	-0.3007	-25.99	-589.4	-0.01286
$(3,3)$	-4.322	1.635	13.16	224.6	0.9691