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Top quark production at threshold with $\mathcal{O}(\alpha_s^2)$ accuracy

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Abstract

We calculate the next-to-next-to-leading order correction to the cross section for top quark pair production in e^+e^- annihilation in the threshold region, resumming all $\mathcal{O}\left[(\alpha_s/\beta)^n \times (\alpha_s^2, \beta^2, \alpha_s\beta)\right]$ terms of perturbation series. We find that the magnitude of the NNLO correction is comparable to the size of the NLO corrections.

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I. INTRODUCTION

The cross section of hadron production in e^+e^- annihilation belongs to the best known quantities in high energy physics. Far away from quark thresholds the cross section is well approximated by the results obtained in perturbative QCD (for a review see [1]). The situation is not so clear at quark thresholds which, however, are known to be of importance for a number of physical applications.

Among such applications, a special place is occupied by a threshold production of $t\bar{t}$ pairs at the Next Linear Collider. It was suggested in Ref. [2], that the large width of the top quark provides a natural cutoff for long–distance effects and, therefore, reliable predictions for the $t\bar{t}$ threshold production cross section are possible in perturbative QCD. Since then, the threshold production cross section of $t\bar{t}$ was studied in great detail [3–7]. The commonly accepted conclusion [6,8] is that one can perform precision studies of various quantities of direct physical interest (top mass, top width, strong coupling constant, etc.), once accurate measurements in the threshold region are conducted. However, all these

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studies were performed using predictions for the top threshold production cross section valid up to $\mathcal{O}(\alpha_s)$ and, therefore, suffered from the ignorance of higher order QCD effects.

It is worth emphasizing, that calculation of radiative corrections to the threshold cross section differs from standard perturbative calculations, which are done for higher energies. The difference is because of the fact, that, close to the threshold, the conventional perturbation theory breaks down [9]. The physical origin of this phenomena is known from quantum mechanics: considering Coulomb potential as a perturbation, one gets series in α/β, where α is the strength of the potential and β is the particle velocity. When the velocity is small, this ratio becomes large and meaningful predictions can be only achieved once the series is resummed. It was demonstrated in [9], that if such resummation is performed, the threshold cross section becomes proportional to the square of the Coulomb wave function at the origin. In Ref. [2] this result was generalized to the situation, when the produced particles are unstable. It was concluded, that the cross section for the quark pair production is proportional to the imaginary part of the non–relativistic Green function of the $\bar{Q}Q$ system, evaluated for complex energies.

Since then, it was also realized, that the $\mathcal{O}(\alpha_s)$ corrections can be easily incorporated, because contributions of soft and hard scales completely factorize with this accuracy. The absence of this factorization property, as well as the technical difficulties with explicit higher order calculations, were the stumbling blocks in achieving the $\mathcal{O}(\alpha_s^2)$ accuracy. It is remarkable, that new results, obtained in the last several years, permit a relatively easy determination of these corrections.

In what follows, we present the calculation of the threshold cross section for the $t\bar{t}$ pair production which is valid with $\mathcal{O}(\alpha_s^2, \alpha_s\beta, \beta^2)$ accuracy.

II. THE FRAMEWORK OF THE CALCULATION

We first discuss a framework of our calculations and introduce all relevant notations. The threshold region is characterized by a small value of the quark velocity β:

$$\beta = \sqrt{1 - \frac{4m^2}{s}} \ll 1. \tag{1}$$

To order $\mathcal{O}(\alpha_s^2, \alpha_s\beta, \beta^2)$, dynamics of slowly moving quark–antiquark pair is governed by a non–relativistic Hamiltonian[1]:

$$H = H_0 + V_1(r) + U(\boldsymbol{p}, \boldsymbol{r}), \tag{2}$$

$$H_0 = \frac{\boldsymbol{p}^2}{m} - \frac{C_F a_s}{r}, \tag{3}$$

$$V_1(r) = -\frac{C_F a_s^2}{4\pi r} \left\{ 2\beta_0 \ln(\mu' r) + a_1 \right.$$

[1]One can describe the $\bar{Q}Q$ system by means of the non–relativistic quantum mechanics to this order since the radiation of real gluons shows up only at $\mathcal{O}(\beta^3)$ order.

$$+\frac{a_s}{4\pi}\left[\beta_0^2\left(4\ln^2(\mu' r)+\frac{\pi^2}{3}\right)+2(\beta_1+2\beta_0 a_1)\ln(\mu' r)+a_2\right]\Bigg\}, \tag{4}$$

$$U(\boldsymbol{p},\boldsymbol{r}) = -\frac{\boldsymbol{p}^4}{4m^3}+\frac{\pi C_F a_s}{m^2}\delta^{(3)}(\boldsymbol{r})-\frac{C_F a_s}{2m^2 r}\left(\boldsymbol{p}^2+\frac{\boldsymbol{r}(\boldsymbol{r}\boldsymbol{p})\boldsymbol{p}}{r^2}\right)$$
$$+\frac{3C_F a_s}{2m^2 r^3}\boldsymbol{S}\boldsymbol{L}-\frac{C_F a_s}{2m^2}\left(\frac{\boldsymbol{S}^2}{r^3}-3\frac{(\boldsymbol{S}\boldsymbol{r})^2}{r^5}-\frac{4\pi}{3}(2\boldsymbol{S}^2-3)\delta^{(3)}(\boldsymbol{r})\right)-\frac{C_A C_F a_s^2}{2mr^2}. \tag{5}$$

In the above equations, the strong coupling constant is evaluated at the scale μ:

$$a_s \equiv \alpha_s(\mu). \tag{6}$$

The scale μ' equals to μe^{γ}, γ is the Euler constant.

The operator $U(\boldsymbol{p},\boldsymbol{r})$ is the QCD generalization of the standard Breit potential [10]. The last term in Eq.(5) is the non–Abelian contribution, originating from a correction to the Coulomb gluon exchange, caused by a magnetic gluon [11]. The potential $V_1(r)$ represents a deviation of the static QCD potential from the Coulomb one. It was calculated to order α_s^2 in [12] and to order α_s^3 in [13]. The coefficients there read explicitly:

$$\begin{aligned}
\beta_0 &= \frac{11}{3}C_A-\frac{4}{3}N_L T_R,\\
\beta_1 &= \frac{34}{3}C_A^2-\frac{20}{3}C_A T_R N_L-4C_F T_R N_L,\\
a_1 &= \frac{31}{9}C_A-\frac{20}{9}T_R N_L,\\
a_2 &= \left(\frac{4343}{162}+6\pi^2-\frac{\pi^4}{4}+\frac{22}{3}\zeta_3\right)C_A^2-\\
&\left(\frac{1798}{81}+\frac{56}{3}\zeta_3\right)C_A T_R N_L-\left(\frac{55}{3}-16\zeta_3\right)C_F T_R N_L+\left(\frac{20}{9}T_R N_L\right)^2.
\end{aligned} \tag{7}$$

For the $SU(3)$ color group, the color factors are $C_A=3, C_F=4/3, T_R=1/2$. $N_L=5$ is the number of quarks whose masses have been neglected.

Given the Hamiltonian H, one can find the Green function for the Schrödinger equation:

$$(H-E-i\delta)G(E;\boldsymbol{r},\boldsymbol{r}')=\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}'). \tag{8}$$

Once the Green function is found, the cross section of the non–relativistic $Q\bar{Q}$ pair production in e^+e^- annihilation[2] is obtained as:

$$\sigma(s)=\frac{4\pi\alpha^2}{3s}R(s), \tag{9}$$

[2]In what follows, we consider only photon mediated process and do not take into account the Z–boson exchange. The axial-vector coupling of the Z–boson contributes $\mathcal{O}(\beta^2)$ relative correction to the threshold cross section. The vector Ze^+e^- coupling is also suppressed, but can be taken into account in the same way as the photon contribution, which we treat in this paper.

where

$$R(s) = \lim_{r\to 0} \mathrm{Im}\left[N_c e_Q^2 \frac{24\pi}{s}\left(1-\frac{\boldsymbol{p}^2}{3m^2}\right) G(E;\boldsymbol{r},0)\right], \qquad E=\sqrt{s}-2m. \tag{10}$$

In Eq.(10), we have included the $\mathcal{O}(\beta^2)$ correction originating from the expansion of the vector current which produces and annihilates a heavy $Q\bar{Q}$ pair in the triplet S–state. The quantity $R(s)$ will be the central object for further discussion.

If a calculation of R will be attempted, one will find, that the Green function at the origin does not exist because there are terms in the Hamiltonian H, which behave as $1/r^n,\ n\geq 2$, for small values of r. The difficulty originates from the fact, that the region $r\to 0$ is not properly treated in the Hamiltonian. Indeed, small values of r correspond to a region in the momentum space, where a typical momentum transfer between $\bar{Q}$ and Q, is of the order of the quark masses and therefore quarks cannot be considered as non–relativistic. For this reason, the use of the Hamiltonian H in actual calculations leads to the divergencies, which appear for $r\to 0$.

The way to circumvent this difficulty is as follows. In order to perform a calculation, one introduces a cutoff λ, such that $\alpha_s m \ll \lambda \ll m$. The momenta region where $k\ll\lambda$ is a non–relativistic region and can be described using a Hamiltonian H. The momenta region with $k\gg\lambda$ is a relativistic one and the calculation in this region should be performed using the rules of quantum field theory. We note that this is rather standard procedure for calculations, related to bound state problems. It is also well known that its practical realization often requires substantial effort.

However, there is a possibility to use the result of the non–relativistic calculation with the cutoff in the following way: one takes the limit of the obtained result, considering kinematic region where $\alpha_s \ll \beta \ll 1$. In this particular region, the non–relativistic results are still valid; on the other hand as long as $\alpha_s/\beta \ll 1$, the resummation of the Coulomb effects is not necessary. Therefore, in this particular region, one can calculate the corrections applying the standard rules of the quantum field theory. In the framework of QCD, such calculations have been performed recently in Ref. [15]. One therefore can match the result of the non–relativistic calculation with the cutoff, directly to the result presented in [15] and in this way completely eliminate the cutoff dependence. This procedure was suggested in [14] and we will call it a direct matching procedure, in accordance with that reference.

In what follows, we pursue this program in QCD. We confine ourselves to a strictly perturbative approach and we do not attempt any discussion of non–perturbative effects. In order to accommodate the phenomenologically relevant case of the unstable top quark, we will consider the total energy E as the complex variable $E\to E+i\Gamma_t$, in the spirit of Ref. [2].

We have to mention at this point that a consistent implementation of the effects related to the finite width of the quark is not attempted in this paper. To $\mathcal{O}(\alpha_s)$ order such effects were studied in [7]. A reliable treatment of these effects to $\mathcal{O}(\alpha_s^2)$ is not available at the moment.

III. CORRECTIONS TO THE GREEN FUNCTION AT THE ORIGIN

Let us first consider the Hamiltonian H_0, Eq.(3), as the leading order Hamiltonian. The corresponding Green function will be denoted by $G_C(\boldsymbol{r},\boldsymbol{r}')$.

We first discuss a correction to the Green function $G_C(\boldsymbol{r},\boldsymbol{r}')$ at the origin, caused by the operator $U(\boldsymbol{p},\boldsymbol{r})$ (see Eq.(5)). The first order correction is:

$$-\int \mathrm{d}^3 r\; G_C(\boldsymbol{r},0)\; U(\boldsymbol{p},\boldsymbol{r})\; G_C(\boldsymbol{r},0). \tag{11}$$

As long as we are interested in the $Q\bar{Q}$ pairs, produced in the triplet S–states, only a corresponding projection of the operator $U(\boldsymbol{p},\boldsymbol{r})$ should be considered. Substituting $\boldsymbol{S}^2 = 2$ and $\boldsymbol{L} = 0$ into Eq.(5), we get:

$$U(\boldsymbol{p},\boldsymbol{r}) = -\frac{\boldsymbol{p}^4}{4m^3} + \frac{11\pi C_F a_s}{3m^2}\delta^{(3)}(\boldsymbol{r}) - \frac{C_F a_s}{2m^2}\left\{\frac{1}{r},\boldsymbol{p}^2\right\} - \frac{C_A C_F a_s^2}{2mr^2}. \tag{12}$$

At this stage, it is advantageous to express this operator in terms of the zeroth order Hamiltonian H_0 in order to apply the equation of motion for the Green function $G_C(\boldsymbol{r},0)$:

$$(H_0 - E)G_C(\boldsymbol{r},0) = \delta^{(3)}(\boldsymbol{r}). \tag{13}$$

This is most easily done by using the commutation relations:

$$[H_0, ip_r] = \frac{4\pi\delta^{(3)}(\boldsymbol{r})}{m} + \frac{2\boldsymbol{L}^2}{mr^3} - \frac{C_F a_s}{r^2}, \tag{14}$$

$$\left\{H_0,\frac{1}{r}\right\} = \frac{2}{r}H_0 + \frac{4\pi\delta^{(3)}(\boldsymbol{r})}{m} + \frac{2}{mr^2}\partial_r, \tag{15}$$

where $p_r = -i(\partial_r + 1/r)$ is the radial momentum operator. Using these relations, one finds that the operator $U(\boldsymbol{p},\boldsymbol{r})$ from Eq.(12) can be presented in the form:

$$U(\boldsymbol{p},\boldsymbol{r}) = -\frac{H_0^2}{4m} - \frac{3C_F a_s}{4m}\left\{H_0,\frac{1}{r}\right\} + \frac{11 C_F a_s}{12m}\left[H_0, ip_r\right] - \frac{(2C_F + 3C_A)\, C_F a_s^2}{6mr^2}. \tag{16}$$

Let us consider the first three terms in Eq.(16). Inserting them into Eq.(11) and using the equation of motion for the Green function (13), we find:

$$\begin{aligned}
&-\int \mathrm{d}^3 r G_C(\boldsymbol{r}',\boldsymbol{r})\left(-\frac{H_0^2}{4m} - \frac{3C_F a_s}{4m}\left\{H_0,\frac{1}{r}\right\} + \frac{11C_F a_s}{12m}[H_0, ip_r]\right) G_C(\boldsymbol{r},\boldsymbol{r}'') \\
&= \left[\frac{\boldsymbol{p}^2}{2m^2} + \frac{C_F a_s}{mr'} + 2ip_r\right] G_C(\boldsymbol{r}',\boldsymbol{r}'') + \int \mathrm{d}^3 r G_C(\boldsymbol{r}',\boldsymbol{r})\left\{\frac{E^2}{4m} + \frac{3C_F a_s E}{2mr}\right\} G_C(\boldsymbol{r},\boldsymbol{r}'').
\end{aligned} \tag{17}$$

We note, that all terms in the above equation, which do not contribute to the imaginary part of the Green function have been omitted.

Leaving aside the "surface" terms in Eq.(17), one sees that the discussed perturbation can be absorbed into the zeroth order equation (13). Therefore, the corresponding correction to $G_C(\boldsymbol{r},\boldsymbol{r}')$ can be taken into account to all orders by finding an exact Green function for the Schrödinger equation

$$(\mathcal{H}-\mathcal{E})\,G(\boldsymbol{r},\boldsymbol{r'}) = \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r'}), \tag{18}$$

with the modified Hamiltonian

$$\mathcal{H} = \frac{\boldsymbol{p}^2}{m} + V(r), \qquad V(r) = -\frac{C_F a_s}{r}\left(1+\frac{3E}{2m}\right) + V_1(r) - \frac{(C_F a_s)^2}{2mr^2}\left(\frac{2}{3}+\frac{C_A}{C_F}\right), \tag{19}$$

and the modified eigenvalue

$$\mathcal{E} = E + \frac{E^2}{4m}. \tag{20}$$

It is clear, that the solution of the above equation will deliver $G(\boldsymbol{r},\boldsymbol{r'})$, which is definitely valid to NNLO accuracy. Moreover, such a solution provides a resummation of some second order corrections.

But there is more important reason to believe that such a treatment is more appropriate for the subthreshold ($\sqrt{s} < 2m$) energy region, than the first order time–independent perturbation theory. Let us consider the energy region below the threshold. For stable quarks, one would observe an appearance of narrow resonances in this region. It is well known, that for the realistic value of the top quark width ($\Gamma_t \sim 1.5$ GeV), the resonances are smeared. Still, the excitation curve exhibits a maximum close to the position of the lowest lying resonance.

For the purpose of discussion, we write an expression for an exact Green function:

$$G(E+i\Gamma;0,0) = \sum_n \frac{\psi_n^2(0)}{E_n - E - i\Gamma} + \int_0^\infty \frac{dk}{2\pi}\frac{\psi_k^2(0)}{E_k - E - i\Gamma}, \tag{21}$$

where $\psi_{n,k}$ and $E_{n,k}$ are exact eigenfunctions and eigenvalues, which correspond to a Hamiltonian H. When we calculate this Green function using time–independent perturbation theory, we expand both numerators and denominators of the above equation in power series. One readily realizes, that this procedure is not so harmless for energy denominators, especially when the energy E is close to the position of the resonance. On the other hand, if the Green function is obtained as the solution of the Schrödinger equation, one gets the result directly in the form of Eq.(21). It is mainly for this reason, that we think it is more safe to solve the Schrödinger equation exactly, than to perform the first order perturbation theory calculations.

It is not so straightforward, however, to obtain a numerical solution for such a problem, since the perturbation $1/r^2$ (cf. Eq.(19)) is too singular at the origin. As was already indicated above, the proper approach is to introduce a cutoff r_0 and to extract all terms which have a non–analytic dependence with respect to r_0. On the other hand, all terms which have extra powers of r_0, so that the limit $r_0 \to 0$ can be taken, will be set to zero. Later on, the non–analytic r_0–dependent terms will be determined by a matching of the result of such calculation to its perturbative counterpart [15]. In the next section we will show, how this procedure can be implemented for the numerical solution of the Schrödinger equation (18).

The "surface" terms from Eq.(17) will be discussed later. We note here, that these terms are linear in $G_C(\boldsymbol{r},\boldsymbol{r'})$ and, therefore, contain at most the first–order poles in energy. Hence they do not contribute to the shift of the energy levels and there is no need to account for them beyond the first order.

IV. NUMERICAL SOLUTION OF THE SCHRÖDINGER EQUATION

In this section, we demonstrate, how numerical solution of the Schrödinger equation with the potential $V(r)$ (cf. Eqs.(18) and (19)) can be constructed. The Schrödinger equation for the S–wave Green function is written as:

$$\left(\frac{-1}{m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]+V(r)-\mathcal{E}\right)G(r,r')=\frac{1}{4\pi r^2}\delta(r-r'). \tag{22}$$

It is convenient to define a new function $g(r,r')=rr'G(r,r')$, so that the Schrödinger equation simplifies:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+m\left[\mathcal{E}-V(r)\right]\right)g(r,r')=-\frac{m}{4\pi}\delta(r-r'). \tag{23}$$

According to the standard rules, the Green function can be written as:

$$g(r,r')=A\left[g_<(r)g_>(r')\theta(r'-r)+g_<(r')g_>(r)\theta(r-r')\right] \tag{24}$$

where $g_{<,>}(r)$ are two independent solutions of the homogeneous Schrödinger equation which satisfy proper boundary conditions at $r=0$ and $r=\infty$, respectively. Also, the constant A is defined by the jump of the derivative of the Green function at $r=r'$.

The solutions $g_<(r)$ and $g_>(r)$ are constructed using two other independent solutions of the Schrödinger equation, $g_\pm(r)$, with a prescribed behavior at the origin [3]. However, since the potential $V(r)$ in the Schrödinger equation is as singular as $1/r^2$, one would have problems with setting the standard [3] boundary conditions for $g_\pm$ at $r=0$. To overcome this difficulty, we extract the leading asymptotic of the functions $g_\pm(r)$ for small values of r

$$g_\pm(r)=(mC_Fa_sr)^{d_\pm}f_\pm(r), \tag{25}$$

where

$$d_\pm=\frac{1}{2}\left(1\pm\sqrt{1-4\kappa}\right),\qquad \kappa=\frac{(C_Fa_s)^2}{2}\left(\frac{2}{3}+\frac{C_A}{C_F}\right). \tag{26}$$

Substituting $g_\pm(r)$ to the Schrödinger equation, one obtains an equation for the function $f_\pm(r)$, which is now free of the $1/r^2$ term:

$$\left\{\frac{1}{m}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2d_\pm}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]+\left[\mathcal{E}+\frac{C_Fa_s}{r}\left(1+c_1\ln(\mu'r)^2+c_2\ln(\mu'r)+c_3\right)\right]\right\}f_\pm(r)=0. \tag{27}$$

The coefficients c_{1-3} can be easily obtained using Eq.(19) and Eq.(4). One derives then the asymptotics of the function $f_\pm(r)$ for $r\to 0$:

$$f(r)\sim 1-mC_Fa_srA(r)+\mathcal{O}(r^2),\qquad A(r)=h_1\ln^2(\mu'r)+h_2\ln(\mu'r)+h_3, \tag{28}$$

$$h_1=\frac{c_1}{2d}, \tag{29}$$

$$h_2=\frac{1}{2d}\left[c_2-c_1\left(2+\frac{1}{d}\right)\right], \tag{30}$$

$$h_3=\frac{1}{2d}\left[1+c_3-c_2\left(1+\frac{1}{2d}\right)+c_1\left(2+\frac{1}{d}+\frac{1}{2d^2}\right)\right]. \tag{31}$$

In the above equations, f and d stand for $f_\pm$ and $d_\pm$, respectively.

There could be some doubts about the validity of the boundary conditions, as derived from above equations, since d_- is of order α_s^2 and therefore the second power of α_s appears in the *denominator*. Without going into explanations at this point, we mention that upon careful inspection the above boundary conditions appear to be absolutely legitimate. Later, we will present more detailed arguments in favor of such conclusion.

The Green function is constructed as follows. The solution $g_<(r)$ is identified with $g_+(r)$. The solution $g_>(r)$ is constructed from the solutions $g_\pm$ in such a way, that the boundary condition at the infinity is satisfied:

$$g_>(r) = g_-(r) + B\, g_+(r), \qquad g_>(r) \to 0, \qquad \text{for } r \to \infty. \tag{32}$$

Therefore, one finds:

$$B = -\lim_{r\to\infty}\left[\frac{g_-(r)}{g_+(r)}\right]. \tag{33}$$

If the potential $V(r)$ is real, the imaginary part of the Green function is proportional to the imaginary part of the coefficient B. However, this is not the case for the present problem: the Coulomb part of the potential $V(r)$ is energy dependent and we consider the energy to be a complex variable. Therefore, the formula for the Green function at the origin should be modified. The modification is however simple. It is obtained in a straightforward way from the available asymptotics of the functions $\psi_\pm(r)$. The result reads

$$\mathrm{Im}\, G(r_0, r_0) = -\frac{m^3 (C_F a_s)^2}{4\pi W} \mathrm{Im}\left\{ (m C_F a_s r_0)^{2d_+ - 2}\, B - A_+(r_0) - A_-(r_0) \right\}, \tag{34}$$

where $W = -mC_F a_s (d_+ - d_-)$ is the Wronskian of two independent solutions of the Schrödinger equation.

Let us comment on the role of the second term in the above equation (34). If the potential $V(r)$ in the Schrödinger equation were real, the $A_\pm(r)$ would be real as well. Taking the imaginary part, one then completely removes the contribution of the second term in Eq.(34). In our case E is a complex variable, and the coefficient h_3 has a non–zero imaginary part. Moreover, this imaginary part is formally of the order of $\Gamma_t/(md_-)$, which should be considered as a contribution of order unity.

However, the limit $\kappa \to 0$ should exist for the Green function constructed as above. Therefore, the purpose of the last term in Eq.(34) is to cancel the $1/d_-$ singularities of the function B, obtained with the boundary conditions presented in Eqs.(29–31). We note in this respect, that, by switching off the logarithm–dependent perturbations in the expression for $V(r)$, one obtains an exactly solvable Hamiltonian, so that the statements made above can be easily verified. The corresponding discussion can be found in Appendix.

Eq.(34) then explicitly demonstrates, that the non–analytic dependence on the cutoff r_0 is indeed extracted and all power corrections with respect to the cutoff are neglected. This precisely corresponds to the desired form of the Green function of the Schrödinger equation. The residual dependence on r_0 is removed using the direct matching procedure as described in the next section.

V. MATCHING AND FINAL RESULT FOR R

In Eq.(34) we have taken into account all corrections to the ratio R which are due to relativistic effects in the quark–antiquark interaction and which can thus be called the dynamic ones. However, we still have to consider kinematic corrections that are: i) $\mathcal{O}(\beta^2)$ correction to s; ii) $\mathcal{O}(\boldsymbol{p}^2/m^2)$ correction to the quark current (cf. Eq.(10)) and, finally, iii) the "surface" terms from Eq.(17).

Using equation of motion for the Green function (18), we obtain the result for R at NNLO, with both types of corrections included[3]:

$$R_{\rm NNLO} = \frac{3}{2} N_c e_Q^2 C_F a_s \left(1 + C_1 C_F \left(\frac{a_h}{\pi}\right) + C_2 C_F \left(\frac{a_h}{\pi}\right)^2\right)$$
$$\times \frac{1}{\sqrt{1-4\kappa}} {\rm Im} \left\{ \left(1 - \frac{5\beta^2}{6}\right) \left[(m C_F a_s r_0)^{\sqrt{1-4\kappa}-1} B - A_+(r_0) - A_-(r_0) \right] \right\}. \qquad (35)$$

Here we have factored out all energy–independent corrections. They are parametrized by the constants C_1 and C_2, which are divergent [4] in the limit $r_0 \to 0$. For this reason we use $a_h = \alpha_s(m)$ as the expansion parameter for these "hard" corrections.

To get rid of the r_0–dependence we use the direct matching procedure, suggested in Ref. [14]. For this we consider $\sqrt{s} > 2m$, set the width of the top quark Γ_t to zero[5] and equate our result (35) to its perturbative counterpart [15] in the kinematic region $\alpha_s \ll \beta \ll 1$, where both are supposed to be valid. We also set $\mu = m$, so that a_s coincides with a_h.

Let us note, that the direct matching procedure fixes the linear combination of C_2 and $\ln(mr_0)$

$$C_2 C_F \left(\frac{\alpha_s}{\pi}\right)^2 - 2\kappa \ln(mr_0). \qquad (36)$$

If we were working strictly to NNLO, this last combination would be the only thing we need for the final result. However, because of the large difference in scales, which govern relativistic and non–relativistic physics, we would like to write Eq.(35) in a factorized form and include an exact dependence on r_0 into the non–relativistic Green function. For this reason, we have to set a factorization scale. We do this by choosing r_0 in such a way, that the correction to the Coulomb Green function due to the $1/r^2$ perturbation in the region $\alpha_s \ll \beta \ll 1$ is given by $\log(\beta)$, without additional constants (see Appendix for more details).

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[3] We note that, strictly speaking, the "surface" terms in Eq. (17) were derived for the Coulomb Green function, but we substitute the "exact" Green function instead of the Coulomb one in our final formulas.

[4] The strongest divergence in these factors is $1/r_0$, so it not obvious *a priori*, that the functional form of Eq.(34) can be preserved in Eq.(35). Upon careful analysis, this turns out to be possible to NNLO, however.

[5] Note, that in this case the functions $A_\pm(r_0)$ drop out from Eq.(35).

Any other choice of r_0 would correspond to other (also legitimate) value of the factorization scale.

A factorized form (35) of our final result makes sense only if a dependence on a choice of the factorization scale is weak. We have checked that changing the value of the cutoff between $r_0/2$ and $2r_0$ for r_0 given by Eq.(37), we obtain small ($\sim 1-3\%$) variation of the resulting values of R.

Therefore, accoring to our choice, we fix the value of the cutoff

$$r_0 = \frac{e^{2-\gamma}}{2m}, \tag{37}$$

and obtain finally:

$$C_1 = -4; \qquad C_2 = C_F C_2^A + C_A C_2^{NA} + T_R N_L C_2^L + T_R N_H C_2^H, \tag{38}$$

where

$$\begin{aligned} C_2^A &= \frac{39}{4} - \zeta_3 + \pi^2 \left(\frac{4}{3}\ln 2 - \frac{35}{18}\right); \\ C_2^{NA} &= -\frac{151}{36} - \frac{13}{2}\zeta_3 + \pi^2 \left(\frac{179}{72} - \frac{8}{3}\ln 2\right); \\ C_2^H &= \frac{44}{9} - \frac{4}{9}\pi^2; \\ C_2^L &= \frac{11}{9}. \end{aligned} \tag{39}$$

Eq.(35) with definitions provided by Eqs.(37 –39) is our final result for the top quark threshold cross section with the NNLO accuracy.

For numerical purposes, we have chosen $m = 175$ GeV and $\Gamma_t = 1.43$ GeV. As an input value for the strong coupling constant we used $\alpha_s(M_Z) = 0.118$. Fig.1 provides our final results for R_{NNLO} as a function of $\sqrt{s} - 2m$ in comparison with LO and NLO results, for three values of the soft scale $\mu = 50,\ 75,\ 100$ GeV. One can see that the NNLO corrections are as large as the NLO ones.

There is also a moderate scale dependence of the NNLO corrections in the vicinity of the resonance peak. The position of the resonance peak appears to be sensitive to the variations in the scale μ on the level ~ 100 MeV. We note in this respect, that the shift of the ground–state energy due to the Breit perturbation is well known (see Ref. [10] and Appendix) and its expected variation with μ is close to this value.

VI. CONCLUSIONS

We have presented a calculation of the next-to-next-to-leading order corrections to the threshold cross section of the top quark pair production in QCD, summing all $\mathcal{O}\left[(\alpha_s/\beta)^n \times (\alpha_s^2, \beta^2, \alpha_s\beta)\right]$ terms of the perturbation series. We have found, that the NNLO effects are quite sizable.

We have also discussed how the numerical solution of the Schrödinger equation with a singular potential can be constructed.

Let us comment on the large size of the NNLO corrections. We have checked that taken separately, both the Breit perturbation and $\mathcal{O}(\alpha_s^3)$ terms from $V_1(r)$, provide comparable contributions of the same sign. When we take them into account simultaneously in the Schrödinger equation, the NNLO contribution gets enhanced by roughly a factor of two, in the vicinity of the resonance.

When we were writing this paper, we received the preprint [16], where the same problem was studied. Our qualitative conclusion about the size of the NNLO corrections agrees with the conclusion reached in [16].

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APPENDIX

In this Appendix, we discuss the construction of the Green function $G(\mathcal{E}; r_0, r_0)$ in the exactly solvable model, which is described by the Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{m} + V(r), \qquad V(r) = -\frac{C_F a_s}{r}(1+c) - \frac{\kappa}{mr^2}, \tag{40}$$

where $\mathcal{E}$ and c are complex parameters.

The Green function is derived following our discussion in Sect. IV. It is convenient to introduce a new variable $z = C_F a_s m r$ for further discussion. The solution of Eq.(27), which satisfies boundary conditions given by Eq.(28), can be written as

$$f_\pm(z) = e^{\frac{iz}{2\nu}} F\left(d_\pm - i\nu(1+c),\ 2d_\pm,\ -\frac{iz}{\nu}\right), \tag{41}$$

where $\nu = C_F a_s / \left(2\sqrt{\mathcal{E}/m}\right)$. Using the asymptotic form of the confluent hypergeometric function for $\mathrm{Re}\, x = \mathrm{Re}(-iz/\nu) \gg 1$,

$$F(a,b,x) \sim \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}, \tag{42}$$

one obtains the coefficient B, using Eqs.(25) and (33):

$$B(\nu) = \frac{i}{\nu}(i\nu)^{2-2d_+} \frac{\Gamma(2d_-)\Gamma(d_+ - i\nu(1+c))}{\Gamma(2d_+)\Gamma(d_- - i\nu(1+c))}, \tag{43}$$

The functions $A_\pm(r)$ reduce now to the constants:

$$A_\pm(r) = \frac{1+c}{2d_\pm}. \tag{44}$$

We therefore arrive at the final expression for the imaginary part of the Green function $\mathrm{Im}G(r_0,r_0)$ for this model (cf. Eq.(34)):

$$\mathrm{Im}G(r_0,r_0) = \frac{m^2 C_F a_s}{4\pi(d_+ - d_-)}\mathrm{Im}\left\{(mC_F a_s r_0)^{2d_+ -2}\, B(\nu) - \frac{1+c}{2\kappa}\right\}. \tag{45}$$

Let us first demonstrate that the proper limit $\kappa \to 0$ exists for the imaginary part of the Green function defined through Eq.(45). In this limit, our model reduces to the ordinary Coulomb problem, so that the Green function in Eq.(45) should give us the imaginary part of the Coulomb Green function at the origin. To see how this happens, we expand the Green function Eq.(45) in power series in κ. The first term in the expansion of $B(\nu)$ is equal to $(1+c)/(2\kappa)$. This term is completely canceled by the last term in Eq.(45). The next term in the expansion provides the r_0–independent result:

$$\lim_{\kappa\to 0} G(r_0,r_0) = \frac{m^2 C_F a_s}{4\pi}\mathrm{Im}H(\nu,\beta),$$

where

$$H(\nu,\beta) = \frac{i}{2\nu} - (1+c)\left[\gamma + \ln(-i\beta) + \psi\left(1 - i\nu(1+c)\right)\right], \qquad \psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z), \tag{46}$$

which exactly coincides with the imaginary part of the Coulomb Green function at the origin.

We then set $c=0$ and expand Eq.(45) up to $\mathcal{O}(\kappa)$ to obtain the correction to the Coulomb Green function due to the $1/r^2$ perturbation. The result can be written as:

$$\mathrm{Im}\left[\delta G(r_0,r_0)\right] = \frac{m^2 C_F a_s \kappa}{4\pi}\mathrm{Im}\left\{H(\nu,\beta)^2 + \frac{\beta^2}{(C_F a_s)^2}\right\}, \tag{47}$$

where the value of r_0 from Eq.(37) has been used. For stable quarks, the last term in Eq.(47) can be disregarded. For unstable quarks, it contributes an $\mathcal{O}(\Gamma_Q/m_Q)$ relative correction to the Green function at the origin, which is beyond the intended accuracy and can be omitted.

Let us also emphasize one advantage of the Green function as obtained from Eq.(45). Consider the stable quark case. Then, for negative energies, the Green function should deliver the first order poles which correspond to the appearance of the $Q\bar{Q}$ bound states in the spectrum. Eq.(45) shows, that such poles are provided by the singularities of the function $\Gamma(d_+ - i\nu(1+c))$. The corresponding eigenvalues of the Hamiltonian (40) are

$$\mathcal{E}_n = -\frac{m(C_F a_s)^2(1+c)^2}{4(n-d_-)^2}. \tag{48}$$

Using the relations (cf. Eqs. (19) and (20))

$$\mathcal{E} = E + \frac{E^2}{4m}, \qquad c = \frac{3E}{2m}, \tag{49}$$

as well as

$$d_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-4\kappa}\right), \qquad \kappa = \frac{(C_F a_s)^2}{2}\left(\frac{2}{3} + \frac{C_A}{C_F}\right), \tag{50}$$

one easily finds that the energy levels are located at

$$E_n \approx -\frac{m\left(C_F a_s\right)^2}{4n^2} + \frac{m\left(C_F a_s\right)^4}{n^3}\left[\frac{11}{64n} - \frac{1}{6} - \frac{C_A}{4C_F}\right]. \tag{51}$$

We note, that this is precisely what one gets, if the energy shift due to the perturbation (5) is calculated using the standard rules of quantum mechanics.

REFERENCES

[1] K.G. Chetyrkin, J.H. Kühn, A. Kwiatkowsky, Phys. Rept. **277** (1996), 189.

[2] V.S. Fadin and V.A. Khoze, JETP Lett. **46** (1987), 525; Sov. J. Nucl. Phys. **48** (1988), 309.

[3] M. Peskin and M. Strassler, Phys. Rev. **D43** (1991), 1500.

[4] W. Kwong, Phys. Rev.**D43** (1991), 1488.

[5] M.Jeźabek, J.H.Kühn and T. Teubner, Z. Physik **C56** (1992), 653;
Y. Sumino, K. Fujii, K. Hagiwara, H. Murayama and C.-K. Ng, Phys. Rev. **D47** (1993), 56.

[6] K. Fujii, T. Matsui and Y. Sumino, Phys. Rev. **D50** (1994), 4341.

[7] K. Melnikov and O. Yakovlev, Phys. Lett. **B324** (1994), 217.
Y. Sumino, Ph.D thesis, Tokyo, 1993;

[8] E. Accomando *et al.*, the ECFA/DESY LC Physics Working Group, *Physics with* e^+e^- *linear colliders*, DESY 97-100, hep-ph/9705442.

[9] A. Sommerefeld, *Atombau und Spektrallinie* (Braunschweig, Vieweg, 1939);
A.D. Sakharov, Zh. Eksp. Teor. Fiz. **18** (1948), 631.

[10] L.D. Landau and E.M. Lifschitz, *Relativistic Quantum Theory*, part 1 (Pergamon, Oxford, 1974).

[11] S.N. Gupta and S.F.Randford, Phys. Rev. **D24** (1981), 2309, (E) *ibid* **D25** (1982), 3430;
S.N. Gupta, S.F.Randford and W.W. Repko, Phys. Rev. **D26** (1982), 3305.

[12] W. Fischler, Nucl. Phys. **B129** (1977), 157;
A. Billoire, Phys. Lett. **B92** (1980), 343.

[13] M. Peter, Nucl. Phys. **B501** (1997), 471.

[14] A.H. Hoang, Phys. Rev. **D56** (1997), 5851.

[15] A. Czarnecki and K.Melnikov, hep-ph/**9712222**, to be published in Phys. Rev. Lett.

[16] A.H. Hoang and T. Teubner, preprint UCSD/PTH 98-01, DESY 98-008, hep-ph/9801397.

FIGURES

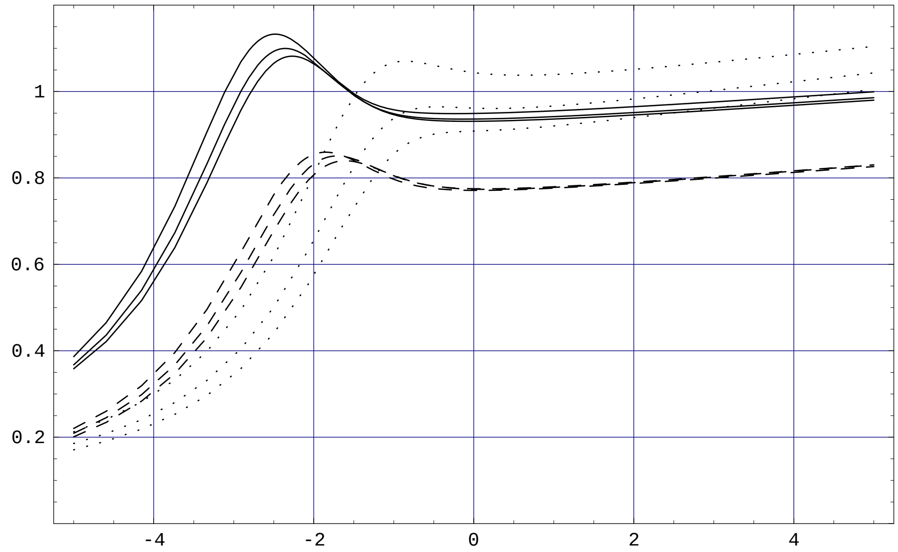

FIG. 1. R_{LO} *(dotted lines),* R_{NLO} *(dashed lines),* R_{NNLO} *(solid lines) as a function of energy* $\sqrt{s}-2m$, *GeV. In all three cases, three curves correspond to different choices of the soft scale* $\mu = 50$ GeV *(upper curves),* $\mu = 75$ GeV *and* $\mu = 100$ GeV *(lower curves). We also use* $m = 175$ GeV, $\Gamma_t = 1.43$ GeV *and* $\alpha_s(M_Z) = 0.118$ *as the input parameters.*