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On the relation between QCD potentials in momentum and position space

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Abstract

We derive a formula which relates the QCD potentials in momentum space and in position space in terms of the β function of the renormalization-group equation for the potential. This formula is used to study the theoretical uncertainties in the potential and in particular in its application to the determination of the pole mass m_b when we use perturbative expansions. We demonstrate the existence of these uncertainties for the Richardson potential explicitly and then discuss the limited theoretical accuracy in the perturbative QCD potential. We conclude that a theoretical uncertainty of m_b much below 100 MeV would not be achievable within perturbative QCD.

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In this article we discuss a relation between the static QCD potential [1] in momentum space

$$V(q) = -C_F\,\frac{4\pi\alpha_V(q)}{q^2}, \tag{1}$$

where for quarks $C_F = 4/3$, and the corresponding potential in position space[1]

$$\bar{V}(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3}\,V(q)\,e^{i\mathbf{q}\cdot\mathbf{r}} = -C_F\frac{\bar{\alpha}_V(1/r)}{r}. \tag{2}$$

Eqs. (1) and (2) are understood as the definitions of the couplings $\alpha_V(q)$ and $\bar{\alpha}_V(1/r)$. In addition, they express the fact that these two functions are related because $\bar{V}(r)$ is the Fourier transform of $V(q)$. We can obtain a useful expression for this functional relation

$$\bar{\alpha}_V(1/r) = F[\alpha_V(q)] \tag{3}$$

using the renormalization-group equation of the coupling $\alpha_V(q)$, and we will study the consequences of this relation.

First we relate the coupling at a general scale, $\alpha_V(q)$, to its value at a specific renormalization scale μ. Let us express $\alpha_V(q)$ as a power series

$$\alpha_V(q) \equiv \Phi(\alpha_V(\mu), t) = \sum_{n=0}^{\infty} c_n(\mu)\,t^n \qquad \text{where} \quad t = \ln(\mu^2/q^2). \tag{4}$$

It follows that

$$\alpha_V(\mu) = \Phi(\alpha_V(\mu), 0) = c_0(\mu). \tag{5}$$

For static heavy quarks, the potential energy is a physical quantity. As a result, $\alpha_V(q)$ obeys the renormalization-group equation

$$\mu^2\frac{d}{d\mu^2}\,\alpha_V(q) = \frac{\partial\Phi(\alpha_V, t)}{\partial t} + \beta_V(\alpha_V)\,\frac{\partial\Phi(\alpha_V, t)}{\partial\alpha_V} = 0, \tag{6}$$

where the β function for α_V is defined by

$$\beta_V(\alpha_V) \equiv \mu^2\frac{\partial\alpha_V(\mu)}{\partial\mu^2} = -4\pi\sum_{n=0}^{\infty}\beta_{V,n}\left(\frac{\alpha_V(\mu)}{4\pi}\right)^{n+2}. \tag{7}$$

Since the coefficients $c_n(\mu)$ in eq. (4) can be expressed as partial derivatives of Φ with respect to t,

$$c_n(\mu) = \frac{1}{n!}\,\frac{\partial^n}{\partial t^n}\,\Phi(\alpha_V(\mu), t)\bigg|_{t=0}, \tag{8}$$

one can show using the renormalization-group equation

$$c_n(\mu) = \frac{1}{n!}\left(-\beta_V(\alpha_V)\,\frac{\partial}{\partial\alpha_V}\right)^n \Phi(\alpha_V, t=0) = \frac{1}{n!}\left(-\beta_V(\alpha_V)\,\frac{\partial}{\partial\alpha_V}\right)^n \alpha_V(\mu). \tag{9}$$

[1] We use the notation $q = |\mathbf{q}|$ and $r = |\mathbf{r}|$.

Combining this expression with eqs. (4) and (8), we find the relation between the couplings $\alpha_V(q)$ and $\alpha_V(\mu)$:

$$\alpha_V(q) = \exp\left[-t\,\beta_V(\alpha_V)\,\frac{\partial}{\partial\alpha_V}\right]\alpha_V(\mu) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(-\beta_V(\alpha_V)\,\frac{\partial}{\partial\alpha_V}\right)^n\alpha_V(\mu). \tag{10}$$

From this expression, one finds

$$c_n(\mu) = \mathcal{O}\left([\alpha_V(\mu)]^{n+1}\right). \tag{11}$$

Now we can perform the Fourier transform in eq. (2) using the following formula:

$$\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,t^n\,\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2} = \left.\frac{\partial^n}{\partial u^n}\,\mathcal{F}(r,\mu,u)\right|_{u=0}, \tag{12}$$

where for $-1 < u < 1/2$,

$$\mathcal{F}(r,\mu,u) \equiv \mu^{2u}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{e^{i\mathbf{qr}}}{(q^2)^{1+u}} = \frac{(\mu r')^{2u}}{4\pi r}\,f(u), \tag{13}$$

with

$$\begin{aligned} f(u) &= \sqrt{\frac{\tan(\pi u)}{\pi u}}\,\exp\left[\sum_{n=1}^{\infty}\frac{(2u)^{2n+1}}{2n+1}\zeta(2n+1)\right] \\ &= \sum_{n=0}^{\infty} f_n\,u^n \\ &= 1+\frac{\pi^2 u^2}{6}+\frac{8\,\zeta(3)\,u^3}{3}+\frac{19\,\pi^4\,u^4}{360}+\left(\frac{4\,\pi^2\,\zeta(3)}{9}+\frac{32\,\zeta(5)}{5}\right)u^5+\cdots \end{aligned} \tag{14}$$

Here,

$$r' = r\,e^{\gamma_E}, \tag{15}$$

$\gamma_E = 0.57721...$ is Euler's constant, and ζ denotes the Riemann ζ function.[2] Higher order terms in the expansion of $f(u)$ at $u=0$ can be easily obtained using e.g. Mathematica [5]. It follows from eqs. (12) and (13) that

$$F[t^n] = \left.\frac{\partial^n}{\partial u^n}\left[e^{uy}\,f(u)\right]\right|_{u=0} = \sum_{k=0}^{n}\binom{n}{k}\,y^k\left.\frac{\partial^{n-k}f(u)}{\partial u^{n-k}}\right|_{u=0} = \sum_{k=0}^{n}\frac{n!}{k!}\,f_{n-k}\,y^k, \tag{16}$$

[2]Different expressions for $\mathcal{F}(r,\mu,u)$ are derived in [3]. One of those,

$$\mathcal{F}(r,\mu,u) = \frac{(\mu r)^{2u}}{4\pi^2 r}\,\frac{\Gamma(\frac{1}{2}+u)\,\Gamma(\frac{1}{2}-u)}{\Gamma(1+2u)},$$

can be used to derive eqs. (13) and (14) via the formulas [4]

$$\Gamma(\tfrac{1}{2}+z)\,\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos(\pi z)}$$

and

$$\ln\Gamma(1+z) = \frac{1}{2}\,\ln\,\frac{\pi z}{\sin(\pi z)} - \gamma_E z - \sum_{n=1}^{\infty}\frac{z^{2n+1}}{2n+1}\,\zeta(2n+1).$$

where

$$y = \ln(\mu^2 r'^2). \tag{17}$$

Then from eq. (10) we obtain

$$\bar{\alpha}_{V}(1/r) = F[\alpha_{V}(q)] \;=\; \sum_{n=0}^{\infty} \frac{F[t^n]}{n!} \left(-\beta_{V}(\alpha_{V})\,\frac{\partial}{\partial\alpha_{V}}\right)^n \alpha_{V}(\mu). \tag{18}$$

In particular, for $y = 0$, which implies $\mu = 1/r'$,

$$F[t^n](y=0) = n!\, f_n. \tag{19}$$

Hence,

$$\bar{\alpha}_{V}(1/r) = \sum_{n=0}^{\infty} f_n \left(-\beta_{V}(\alpha_{V})\,\frac{\partial}{\partial\alpha_{V}}\right)^n \alpha_{V}(q = 1/r'). \tag{20}$$

This equation is the specific representation of the functional relation (3) which we set out to derive. The function $\beta_{V}(\alpha_{V})$ is defined in eq. (7) and the coefficients f_n in eq. (14).

Before studying consequences of eq. (20) for the perturbative QCD potential, let us discuss first a model which we can solve numerically and which describes the energy levels of quarkonia very well. The model that we consider is the Richardson potential [6] which in momentum space reads:

$$V_{\rm R}(q) \;=\; -4\pi C_F \frac{\alpha_{V}^{(R)}(q)}{q^2} \;=\; -\frac{16\pi^2\, C_F}{\beta_0\, q^2 \ln(1+q^2/\Lambda_{\rm R}^2)}\;. \tag{21}$$

A very good description of the charmonium and bottomonium states is obtained using this potential for $\Lambda_{\rm R} = 0.4$ GeV and

$$\beta_0 = 11 - \frac{2}{3} n_f, \tag{22}$$

where $n_f = 3$ is the number of the light quarks. In Fig.1 the function $\alpha_{V}^{(\rm R)}(q)$ is plotted as the dash–dotted line for q between 1 and 10 GeV. The next-to-next-to-leading order (three-loop renormalization-group improved) QCD coupling $\alpha_{V}^{(4)}(q)$ [3] for $n_f = 4$ active flavours and $\alpha_{\overline{MS}}(m_Z)$= 0.119 is also shown as the solid line.[3] It is seen that in the range of q relevant for the position of $\Upsilon(1S)$ the two curves are fairly close. The QCD potential is more attractive which implies that the value of m_b extracted from the mass of $\Upsilon(1S)$ is slightly larger for the next-to-next-to-leading order QCD than for the Richardson potential. In fact in a recent article [8] $m_b = 4.96$ GeV is obtained[4] with an uncertainty of about 100 MeV arising from the uncertainty in the input value of $\alpha_{\overline{MS}}(m_Z)$. For the Richardson potential $m_b = 4.88$ GeV is obtained. All these attractive features suggest that at scales of a few GeV the model closely resembles the true QCD potential.

[3] We used the three-loop renormalization-group equation to evolve the $\overline{MS}$ coupling, and Eq. (9) of [7] without the a^3-term to match the 4-flavour to the 5-flavour theory at the matching scale 5 GeV, and m_b(pole) = 4.88 GeV. The result is not very sensitive to varying this matching scale.

[4] This value is different from what [8] gives as a final value of m_b. It corresponds to the result from the two-loop static QCD potential, which does not include a shift in m_b of about 50 MeV due to relativistic and leading non-perturbative corrections.

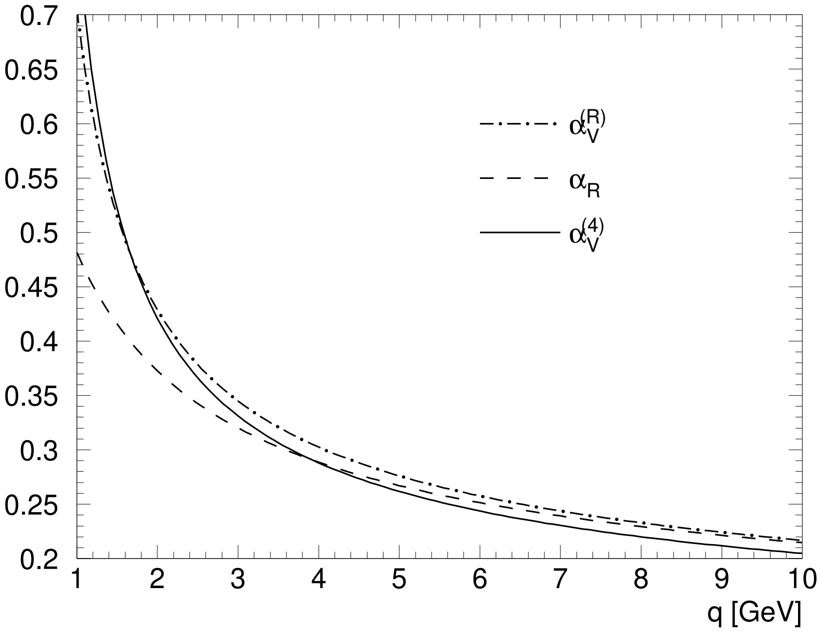

Figure 1: Comparison of the momentum-space couplings: $\alpha_V^{(4)}(q)$ – solid, $\alpha_V^{(\mathrm{R})}(q)$ – dash–dotted, and $\alpha_{\mathrm{R}}(q)$ – dashed line.

For this model, we study the validity of the relation (20) when it is expressed as a perturbative series of α_V in the asymptotic region where the coupling is small. In order to define the perturbative coupling of this model unambiguously in both momentum and position spaces, we subtract the confining part of the Richardson potential

$$V_{\mathrm{R}}(q) = V_{\mathrm{conf}}(q) - 4\pi C_F \frac{\alpha_{\mathrm{R}}(q)}{q^2}. \tag{23}$$

The confining part $V_{\mathrm{conf}}(q)$ corresponds to a linear confining potential in position space

$$\bar{V}_{\mathrm{conf}}(r) = \frac{2\pi C_F}{\beta_0}\Lambda_{\mathrm{R}}^2 r. \tag{24}$$

The coupling

$$\alpha_{\mathrm{R}}(q) = \frac{4\pi}{\beta_0}\left[\frac{1}{\ln(1+q^2/\Lambda_{\mathrm{R}}^2)} - \frac{\Lambda_{\mathrm{R}}^2}{q^2}\right] \tag{25}$$

is considered as a perturbative coupling in momentum space. The coupling $\alpha_{\mathrm{R}}(q)$ is plotted as the dashed line in Fig.1. The corresponding coupling in position space is given by the following

$1/r$ [GeV]	N	$\bar{\alpha}_{R,N}(1/r)$	$\bar{\alpha}_{R}(1/r)$	$\delta\bar{V}_-$ [GeV]	$\delta\bar{V}_+$ [GeV]
10	3	0.2856	0.2714	-0.193	0.189
20	3	0.22262	0.22183	-0.381	0.021
50	4	0.17647	0.17470	-0.222	0.118
10^2	4	0.14868	0.14912	-0.059	0.183
10^3	6	0.098937	0.099017	-0.106	0.120
10^4	9	0.07413536	0.07413084	-0.143	0.060
10^5	11	0.05938222	0.05938213	-0.175	0.012
10^6	12	0.04958339	0.04958315	-0.409	0.318
10^7	12	0.04257994	0.04257992	-1.008	0.185

Table 1: Comparison of the coupling $\bar{\alpha}_{R}(1/r)$ for the Richardson potential and the best approximation obtained by truncating the asymptotic series (29) at $n = N$.

expression [6]:

$$\bar{\alpha}_{\mathrm{R}}(1/r) = \frac{2\pi}{\beta_0}\left[1 - 4\int_1^\infty \frac{du}{u}\frac{e^{-\Lambda_{\mathrm{R}} r u}}{\ln^2(u^2-1)+\pi^2}\right] \tag{26}$$

which can be calculated numerically to a very high precision. In particular the accuracy of our numerical calculations was 10^{-10}. From eq. (25) we calculate the β function

$$\beta_{\mathrm{R}}(\alpha_{\mathrm{R}}) = q^2\frac{\partial\alpha_{\mathrm{R}}(q)}{\partial q^2} = -\frac{\beta_0}{4\pi}\alpha_{\mathrm{R}}^2 + \frac{\beta_0}{4\pi}\frac{[\frac{4\pi}{\beta_0}-\alpha_{\mathrm{R}}]^2}{1+q^2/\Lambda_{\mathrm{R}}^2}, \tag{27}$$

where $q^2/\Lambda_{\mathrm{R}}^2$ is understood as a function of α_{R}. For $q \gg \Lambda_{\mathrm{R}}$ the second term in the expression for β_{R} is negligible. Then our problem becomes very simple: there is only one term in the sum in eq. (7) corresponding to

$$\beta_{\mathrm{R},0} = \beta_0. \tag{28}$$

Instead of eq. (20) we have a much simpler relation

$$\bar{\alpha}_{\mathrm{R}}(1/r) \sim \sum_{n=0}^{\infty} \alpha_{\mathrm{R}}(q)\left(\frac{\beta_0\,\alpha_{\mathrm{R}}(q)}{4\pi}\right)^n n!\,f_n, \tag{29}$$

with $q = 1/r'$. The symbol "$\sim$" indicates that in our derivation non-perturbative higher twist contributions to β_{R} (terms suppressed by powers of $e^{-4\pi/(\beta_0\alpha_{\mathrm{R}})}$) have been neglected. Strictly speaking, the relation (29) may be valid only for $q \gg \Lambda_{\mathrm{R}}$ i.e. for small α_{R}. We have computed the first twelve coefficients f_n and checked that all are positive. For a given $1/r'$, the terms in (29) decrease with increasing n for $n < n_{\mathrm{min}}$ and increase for $n > n_{\mathrm{min}}$. The value n_{min} corresponding to the minimal contribution increases with growing $1/r'$. The best approximation is obtained when the series in (29) is truncated at $N \simeq n_{\mathrm{min}}$. In Table 1 the values are given of the functions $\bar{\alpha}_{\mathrm{R}}(1/r)$ computed numerically from eq. (26) and $\bar{\alpha}_{\mathrm{R,N}}(1/r)$ obtained from the asymptotic expansion (29) truncated at $n = N$. The coefficient β_0 is evaluated for $n_f = 3$ and $1/r$ is varied between 10 and 10^7 GeV. It is evident that for large values of $1/r$ a very good approximation is obtained using the truncated series $\bar{\alpha}_{\rm R,N}(1/r)$. However, at $1/r = 10$ GeV, $n_{\min} = 1$ and the quality of approximation practically does not improve when instead of

$$\bar{\alpha}_{\rm R}(1/r) \simeq \alpha_{\rm R}(q = 1/r') \tag{30}$$

the formula including the $\mathcal{O}(\alpha_{\rm R}^3)$ term is used. At $1/r = 20$ GeV inclusion of the cubic term improves the quality of approximation, and at $1/r = 100$ GeV also the term $\mathcal{O}(\alpha_{\rm R}^4)$ is needed. Let the truncution at $n = n_+$ result in the closest approximation of $\bar{\alpha}_{\rm R}$ from above and $\delta\bar{V}_+(r)$ denote the difference between the exact and the approximate values of the potential $\bar{V}_{\rm R}(r)$. If the contribution of the term for $n = n_+$ is not included, i.e. the series in (29) is truncated at $n = n_+ - 1$ the closest approximation from below is obtained. Let $\delta\bar{V}_-(r)$ be the corresponding difference of the potentials. The values of the functions $\delta\bar{V}_\pm(r)$ are also given in Table 1. One can estimate the theoretical uncertainty of the perturbative approach by considering the change in the value of the potential

$$\delta\bar{V}(r) = \delta\overline{V}_+ - \delta\overline{V}_- \tag{31}$$

corresponding to truncating the asymptotic formula at $n = n_+ - 1$ and $n = n_+$. It is interesting[5] that $\delta\bar{V}(r)$ does not change drastically with r and is of the order of $\Lambda_{\rm R}$.

In perturbative calculations of the energy levels of a $Q\bar{Q}$ system in this model, the truncation of the asymptotic series for the potential leads to an uncertainty in the perturbative pole mass m_Q of the heavy quark. This uncertainty is of the order of $\delta m \sim \delta\bar{V}(r_0)/2$, where r_0 denotes the typical size of the bound-states. This estimation is obtained assuming that the shift in the binding energy due to the change of the coupling $\bar{\alpha}_{\rm R}$ is equal to $-\,\delta\bar{V}(r_0)$, which would be the case for the Coulomb potential. The shift in the binding energy is then compensated by shifting the masses of Q and $\bar{Q}$ by δm. One may think that for a stable quark Q the uncertainty in m_Q can be reduced by comparing the $Q\bar{Q}$ bound states of different sizes, thereby disentangling the correlation in the dependences on the coupling and mass. However, as it has already been mentioned the variation of $\delta\bar{V}(r)$ with r is quite moderate. Moreover at larger distances non-perturbative effects become important. So, in fact it is difficult to reduce the uncertainty in the mass determination significantly.[6]

It may appear too pessimistic to estimate δm from $\delta\bar{V}$. One can argue that instead of considering the truncated asymptotic series for $\alpha_{\rm R}$ the exact expression (26) should be used in the Schrödinger equation for the $Q\bar{Q}$ system. Then m_Q can be determined from the mass of the $Q\bar{Q}$ bound-states without any theoretical uncertainty. This argument is based, however, on the complete knowledge of the function $\alpha_{\rm R}(q)$ including the range of its argument where non-perturbative contributions are very important. In purely perturbative calculations δm has to

[5]This can be shown easily for a small $\alpha_{\rm R}$: Since asymptotically $f_n \sim 2^n$, it follows that $n_{\min} \sim 2\pi/(\beta_0\alpha_{\rm R}) \sim \ln(1/(\Lambda_{\rm R} r))$. Then one can estimate the size of the last term included or rejected in eq. (29) to be $\delta\bar{\alpha}_{\rm R} \sim \Lambda_{\rm R} r$, which leads to $\delta\bar{V}(r) \sim \Lambda_{\rm R}$.

[6]Behind this argument for the Richardson potential lies a following corresponding perspective in QCD. In principle one is able to determine the two independent fundamental parameters of QCD, m_Q and α_s, from a sufficient number of different physical observables $\mathcal{O}_i(m_Q, \alpha_s)$ by exploiting their different dependences on these parameters. When we express the observables in perturbative expansions of α_s, however, we encounter uncertainties of the order of $\Lambda_{QCD} \sim 200$ MeV originating from truncations of the asymptotic series. Then, if we try to extract the values of m_Q and α_s from these observables, the uncertainties should be attached to both of these paramters which cannot be reduced within a purely perturbative approach.

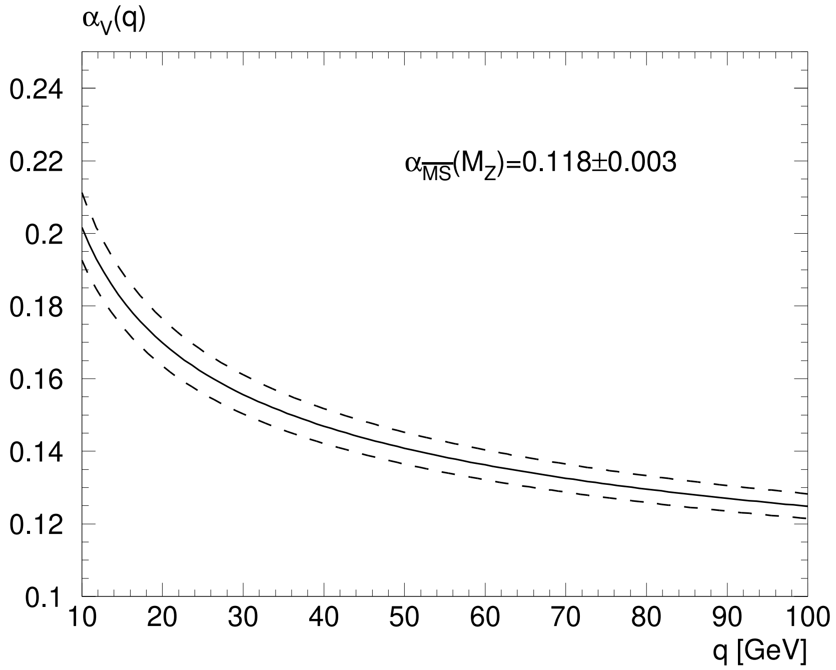

Figure 2: The momentum-space couplings $\alpha_V^{(5)}(q)$ for different values of $\alpha_{\overline{MS}}(m_Z)$.

be interpreted as a perturbative theoretical uncertainty in m_Q due to the asymptotic character of the series (29).

It seems reasonable to expect that the qualitative features of the asymptotic expansion (20) should be similar for the Richardson potential and for the QCD potential. This is a useful assumption because only limited information is available in the latter case. In perturbative QCD the coupling α_V is expressed as an asymptotic series in $\alpha_{\overline{MS}}$ and only the first three terms are known [2, 3]. In Fig. 2, $\alpha_V(q)$ is shown for $n_f = 5$ active flavours and q between 10 and 100 GeV. The solid line corresponds to $\alpha_{\overline{MS}}(m_Z) = 0.118$ and the dashed lines to changing this input value by ± 0.003 which is the present error of this parameter [9]. In QCD the first three coefficients of the β function (7) are also known [10, 2, 3]:

$$
\begin{aligned}
\beta_{V,0} &= \beta_0, \\
\beta_{V,1} &= \beta_1 = 102 - \frac{38}{3} n_f, \\
\beta_{V,2} &= 1854 - \frac{2239}{6} n_f + \frac{377}{54} {n_f}^2 + \pi^2 \left(18 - \frac{3\,\pi^2}{4}\right) (33 - 2\,n_f) \\
&\quad + \left(726 - \frac{704}{3} n_f + \frac{104}{9} {n_f}^2\right) \zeta(3).
\end{aligned}
\tag{32}
$$

Using this information one can write a formula for $\bar{\alpha}_V(1/r)$ which is accurate up to fifth order in α_V:

$$\bar{\alpha}_V = \alpha_V \Big\{1 + 2\,\beta_0{}^2 f_2\, \mathrm{a_v}{}^2 + \Big(5\,\beta_0\,\beta_1\, f_2 + 6\,\beta_0{}^3\, f_3\Big)\, \mathrm{a_v}{}^3 + \Big[\Big(3\,\beta_1{}^2 + 6\,\beta_0\,\beta_{V,2}\Big)\, f_2 + 26\,\beta_0{}^2\,\beta_1\, f_3 + 24\,\beta_0{}^4\, f_4\Big]\, \mathrm{a_v}{}^4\Big\} + \mathcal{O}(\alpha_V^6), \quad (33)$$

where $\alpha_V \equiv \alpha_V(q = 1/r')$ and $\mathrm{a_v} = \alpha_V/(4\pi)$. Let us remark that the term quadratic in α_V is absent because it has been absorbed by shifting the argument of the coupling α_V. After simple algebra the following formula is derived:

$$\begin{aligned}\bar{\alpha}_V = \alpha_V \Bigg\{ & 1 + \left(11 - \frac{2}{3}\,n_f\right)^2 \frac{\pi^2}{3}\, \mathrm{a_v}{}^2 + \\ & \left[\left(935 - \frac{1555}{9} n_f + \frac{190}{27} n_f{}^2\right)\pi^2 + \left(21296 - 3872 n_f + \frac{704}{3} n_f{}^2 - \frac{128}{27} n_f{}^3\right)\zeta(3)\right]\mathrm{a_v}{}^3 + \\ & \Bigg[\left(25596 - \frac{39797}{6} n_f + \frac{21913}{54} n_f{}^2 - \frac{377}{81} n_f{}^3\right)\pi^2 + \Bigg(\frac{376189}{15} - \frac{237952}{45} n_f + \\ & \frac{19472}{45}\, n_f{}^2 - \frac{6688}{405}\, n_f{}^3 + \frac{304}{1215}\, n_f{}^4\Bigg)\pi^4 + \left(-\frac{1089}{4} + 33\, n_f - n_f{}^2\right)\pi^6 \\ & + \left(855712 - \frac{1889888}{9}\, n_f + \frac{432640}{27}\, n_f{}^2 - \frac{31616}{81}\, n_f{}^3\right)\zeta(3) \\ & + \left(7986 - \frac{9196}{3}\, n_f + \frac{2552}{9}\, n_f{}^2 - \frac{208}{27}\, n_f{}^3\right)\pi^2\,\zeta(3)\Bigg]\mathrm{a_v}{}^4 + \mathcal{O}(\mathrm{a_v}{}^5)\Bigg\}. \end{aligned} \quad (34)$$

The numerical values of the coefficients in the above expansions for $n_f = 5$ are equal to:

$$\bar{\alpha}_V = \alpha_V + 1.22454\,\alpha_V{}^3 + 5.59618\,\alpha_V{}^4 + 32.2015\,\alpha_V{}^5 + \mathcal{O}(\alpha_V{}^6), \quad (35)$$

whereas for $n_f = 4$ and $n_f = 3$ one obtains

$$\bar{\alpha}_V = \alpha_V + 1.44676\,\alpha_V{}^3 + 7.38182\,\alpha_V{}^4 + 46.3717\,\alpha_V{}^5 + \mathcal{O}(\alpha_V{}^6) \quad (36)$$

$$\bar{\alpha}_V = \alpha_V + 1.68750\,\alpha_V{}^3 + 9.45282\,\alpha_V{}^4 + 64.0389\,\alpha_V{}^5 + \mathcal{O}(\alpha_V{}^6) \quad (37)$$

respectively. The numerical coefficients in the relations (35)-(37) are large. The asymptotic character of the expansion is evident and only for very short distances ($1/r \gtrsim 30$ GeV) all five terms should be kept. At such distances the number of active flavours is $n_f = 5$. The $\mathcal{O}(\alpha_V^4)$ and $\mathcal{O}(\alpha_V^5)$ terms in (35) are equal for $\alpha_V = 0.1738$ and their contributions to $\bar{\alpha}_V$ are 0.0051. This implies that the truncation of the asymptotic series leads to an uncertainty of about 2.8% in $\bar{\alpha}_V$. The value of α_V corresponds to $1/r = 32$ GeV, a distance which is not very different from those probed in $t\bar{t}$ production near the threshold. Phenomenological consequences of the above observation are discussed elsewhere [11]. At the distances probed by $b\bar{b}$ states, the corresponding values of α_V are so large that the quartic term in the expansion (34) must be rejected. Thus it would be simply inconsistent to keep other contributions of this order like those in the relation between α_V and $\alpha_{\overline{MS}}$ even if this relation were known. Our analysis leads to a rather surprising conclusion that in the framework of purely perturbative QCD, theoretical uncertainties cannot be reduced below the present level ! (This is consistent with the discussion given in [8].) If we accept such a radical point of view, we can proceed even further by observing that the cubic term in (34) depends only on β_0 and is the same for the QCD static potential and for the Richardson potential. The latter case suggests that it is not obvious at all if the inclusion of the cubic term improves the accuracy of the formula for $\bar{\alpha}_V$. It may well be that a more precise answer is obtained if the cubic term is also rejected. Without extra information beyond the purely perturbative approach, we cannot answer the question: to reject or not to reject. Clearly, keeping the cubic term implies a stronger attraction between the quarks. Thus, if m_b is determined from the energy of the $b\bar{b}$ ground state, the value of m_b obtained using a perturbative calculation and the potential including the cubic term is larger than the value corresponding to rejecting this term. The difference in m_b is of the order of 100 MeV and our analysis indicates that attempts to reduce this error to a much smaller value within perturbative QCD may be inconsistent.

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References

[1] L. Susskind, in *Weak and Electromagnetic Interactions at High Energy*, R. Balian and C.H. Llewellyn–Smith (eds.), North-Holland, Amsterdam 1977.

[2] W. Fischler, *Nucl. Phys.* **B129** (1977) 157;
A. Billoire, *Phys. Lett.* **B92** (1980) 343.

[3] M. Peter, *Phys. Rev. Lett.* **78** (1997) 602; *Nucl. Phys.* **B501** (1997) 471.

[4] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press Inc., New York 1965.

[5] *Mathematica, A System for Doing Mathematics by Computer*, Wolfram Research, Inc.

[6] L. Richardson, *Phys. Lett.* **B82** (1979) 272.

[7] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Phys. Rev. Lett.* **79** (1997) 2184.

[8] A. Pineda and F.J. Ynduráin, preprint UB–ECM–PF–97-34, hep-ph/9711287.

[9] Review of Particle Physics, R.M. Barnett, et al., *Phys. Rev.* **D54** (1996) 1.

[10] H.D. Politzer, *Phys. Rev. Lett.* **30** (1973) 1346; D.J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30** (1973) 1343;
W.E. Caswell, *Phys. Rev. Lett.* **33** (1974) 244; D.R.T. Jones, *Nucl. Phys.* **B75** (1974) 531; O.V. Tarasov, A.A. Vladimirov and A.Yu. Zharkov, *Phys. Lett.* **B93** (1980) 429; S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B303** (1993) 334.

[11] M. Jeżabek, J.H. Kühn, M. Peter, Y. Sumino and T. Teubner, preprint DESY 98-019, hep-ph/9802373.