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A theory driven analysis of the effective QED coupling at M_Z

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Abstract

An evaluation of the effective QED coupling at the scale M_Z is presented. It employs the predictions of perturbative QCD for the cross section of electron positron annihilation into hadrons up to order α_s^2, including the full quark mass dependence, and of order α_s^3 in the high energy region. This allows to predict the input for the dispersion relations over a large part of the integration region. The perturbative piece is combined with data for the lower energies and the heavy quark thresholds. The result for the hadronic contribution to the running of the coupling $\Delta\alpha_{\rm had}^{(5)}(M_Z^2) = (277.5 \pm 1.7) \times 10^{-4}$ leads to $(\alpha(M_Z^2))^{-1} = 128.927 \pm 0.023$. Compared to previous analyses the uncertainty is thus significantly reduced.

The evolution of the electromagnetic coupling from its definition at vanishing momentum transfer to its value at high energies constitutes the dominant part of radiative corrections for electroweak observables. The accurate determination of $\alpha(M_Z^2)$ is thus essential for any precise test of the theory. At the same time the indirect determination of the masses of heavy, hitherto unobserved particles, e.g. the Higgs boson or SUSY particles, depends critically on this parameter. Of particular importance in this context is the hadronic vacuum polarization. It is nearly as large as the leptonic contribution but can only be related through dispersion relations to the cross section for hadron production in electron positron annihilation, or more conveniently to the familiar R ratio. The integrand can thus be obtained from data, phenomenological models and/or perturbative QCD (pQCD), whenever applicable. A detailed analysis based on data and employing pQCD above 40 GeV has been performed in [1] and their result has been confirmed by subsequent studies [2, 3, 4] following very similar strategies. Alternatively one may employ pQCD also for lower energies, eventually as low as 2 GeV as long as one stays away from the quark thresholds. A first step in this direction has been made in [5]. There, however, only the massless approximation was employed for the normalization of the data. Recently the $\mathcal{O}(\alpha_s^2)$ corrections for R, including the full quark mass dependence, became available [6, 7, 8] which allows to extend pQCD down relatively close to the respective thresholds for charm and bottom production. These results have been used in [9] to evaluate the perturbative contribution to the vacuum polarization. In this work we complete the evaluation by incorporating those contributions which cannot be obtained from pQCD: from the low energy region below roughly 2 GeV, the charmonium and bottomonium resonances and from the charm threshold. The (re)normalization of the data from the PLUTO [10], DASP [11], and MARK I collaborations [12] will be based on the requirement that they agree with pQCD for $\sqrt{s} \leq 3.7$ GeV and $\sqrt{s} \geq 5$ GeV. A similar analysis has been recently performed in [13] which is also based on pQCD, in particular the results of [7] and which, furthermore, provides additional justification for the applicability of pQCD at very low energies around 2 GeV. We will comment on the differences between this analysis and [13], whenever appropriate, below.

Let us briefly describe the theoretical input for our evaluation. The hadronic contribution $\Delta\alpha_{\rm had}^{(5)}$ to the running from the static limit to M_Z is given by

$$\Delta\alpha_{\rm had}^{(5)}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi}\,\mathrm{Re}\int_{4m_\pi^2}^{\infty} \mathrm{d}s\,\frac{R(s)}{s\,(s-M_Z^2-i\epsilon)}\,, \tag{1}$$

where the superscript "(5)" indicates that the top quark is not included in the integral. For the analysis we use $\alpha = \alpha(0) = 1/137.0359895$ and $M_Z = 91.187$ GeV. It leads, after resummation of the leading logarithms, to the following shift

$$\alpha(s) = \frac{\alpha(0)}{1-\Delta\alpha_{\rm lep}(s)-\Delta\alpha_{\rm had}^{(5)}(s)-\Delta\alpha_{\rm top}(s)}\,. \tag{2}$$

The quantity $R(s)$ can be experimentally determined through a measurement of the total cross section for electron positron annihilation into hadrons. From the theoretical side it is defined through the absorptive part of the electromagnetic current correlator

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\,\Pi(q^2) = i\int dx\,e^{iqx}\langle 0|T j_\mu(x) j_\nu(0)|0\rangle\,, \tag{3}$$

$$R(s) = 12\pi\,\mathrm{Im}\,\Pi(q^2 = s+i\epsilon)\,. \tag{4}$$

It can be calculated in the framework of pQCD up to order α_s^3 if quark masses are neglected [14] and up to $\mathcal{O}(\alpha_s^2)$ with full quark mass dependence [6, 7, 9]. In this work pQCD will be assumed to be valid above 1.8 GeV (alternatively 2.125 GeV). In view of the validity of pQCD in tau lepton decays not only for the total rate but also for the spectral function toward the upper end [15] the substitution of inprecise data by pQCD seems justified. Additional support for this approach can also be drawn from the analysis of data for $R(s)$ below 1.8 GeV [13]. pQCD has also been used in [16] in the present context even down to 1.4 GeV. The specific choice of 1.8 GeV (or 2.125 GeV) for the matching between data and theory is dictated by the available data analysis [4, 13] which we adopt for the present purpose.

Also excluded from this theory driven treatment are the threshold region for charmed mesons (the interval from 3.7 GeV to 5 GeV) and, similarly, for bottom mesons (10.5 GeV to 11.2 GeV), and the narrow charmonium and bottomonium resonances, where we use the currently available data. Perturbative QCD is even applicable in the charm and bottom threshold regions, as far as the light quark contributions are concerned. In the bottom threshold region we will therefore use data for the bottom contribution only.

In the perturbative regions one receives contributions from light (u, d and s) quarks whose masses are neglected throughout, and from massive quarks which demand a more refined treatment. Below the charm threshold the light quark contributions are evaluated in order α_s^3 plus terms of order $\alpha_s^2\, s/(4M_c^2)$ from virtual massive quark loops. Above 5 GeV the full M_c dependence is taken into account up to order α_s^2, and in addition the dominant cubic terms in the strong coupling are incorporated, as well as the corrections from virtual bottom quark loops of order $\alpha_s^2\, s/(4M_b^2)$. Above 11.2 GeV the same formalism is applied to the massive bottom quarks and charmed mass effects are taken into account through their leading contributions in an M_c^2/s expansion. All formulae are available for arbitrary renormalization scale μ which allows to test the scale dependence of the final answer. This will be used to estimate the theoretical uncertainties from uncalculated higher orders. Matching of α_s between the treatment with $n_f = 3$, 4 and 5 flavours is performed at the respective threshold values. The influence of the small $\mathcal{O}(\alpha_s^3)$ singlet piece which prevents a clear separation of contributions from different quark species can safely be ignored for the present purpose. The details of the formalism can be found in [9].

In Tab. 1, adopted from [9], the perturbative hadronic contributions are listed separately for a variety of energy intervals. As our default values we adopt $\mu^2 = s$, $\alpha_s^{(5)}(M_Z^2) = 0.118$, $M_c = 1.6$ GeV and $M_b = 4.7$ GeV. In separate columns we list the variations with a change in the renormalization scale, the strong coupling constant and the quark masses:

$$\delta\alpha_s \;=\; \pm 0.003, \quad \delta M_c \;=\; \pm 0.2\ GeV, \quad \delta M_b \;=\; \pm 0.3\ GeV. \tag{5}$$

In principle the theoretical tools are available to include in the perturbative analysis the QED corrections of order α and even $\alpha\alpha_s$. The relative size of the dominant correction is estimated as

$$\frac{\sum_i Q_i^4}{\sum_i Q_i^2}\frac{3}{4}\frac{\alpha}{\pi} \;\approx\; (0.6 - 0.7) \times 10^{-3}\,, \tag{6}$$

and is also included in Tab. 1.

Perturbative QCD is clearly inapplicable in the charm threshold region between 3.7 and 5 GeV where rapid variations of the cross section are observed. Data have been taken more than 15 years ago by the PLUTO [10], DASP [11], and MARK I collaborations [12]. The systematic errors of 10 to 20 % exceed the statistical errors significantly and are reflected in a sizeable spread of the experimental results. To arrive at a reliable evaluation of the charm contribution from this region, we adjust the normalization of the data (for each

Energy range (GeV)	central value	$\delta\mu$	$\delta\alpha_s$	δM_c	δM_b
$1.800 - 2.125$	5.67	0.22	0.04	0.00	0.00
$2.125 - 3.000$	11.66	0.21	0.06	0.01	0.00
$3.000 - 3.700$	7.03	0.06	0.03	0.00	0.00
$1.800 - 3.700$	24.36	0.48	0.13	0.01	0.01
$5.000 - 5.500$	5.44	0.03	0.03	0.06	0.00
$5.500 - 6.000$	4.93	0.03	0.02	0.04	0.00
$6.000 - 9.460$	25.45	0.11	0.08	0.10	0.00
$9.460 - 10.520$	5.90	0.02	0.01	0.01	0.00
$10.520 - 11.200$	3.48	0.01	0.01	0.00	0.00
$5.000 - 11.200$ (without $b\bar{b}$)	45.20	0.19	0.15	0.21	0.01
$11.200 - 11.500$	1.63	0.00	0.01	0.00	0.00
$11.500 - 12.000$	2.62	0.00	0.01	0.00	0.00
$12.000 - 13.000$	4.93	0.01	0.01	0.00	0.00
$13.000 - 40.000$	72.92	0.08	0.12	0.02	0.02
$12.000 - 40.000$	77.85	0.09	0.14	0.02	0.02
$40.000 - \infty$	42.67	0.03	0.06	0.00	0.00
$11.200 - \infty$	124.77	0.12	0.21	0.03	0.02
$1.8 - \infty$ (pQCD)	194.33	0.79	0.49	0.24	0.03
QED	0.11	–	–	–	–

Table 1: Contributions to $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ (in units of 10^{-4}) from the energy regions where pQCD is used (adopted from [9]). For the QED corrections the same intervals have been chosen. For the variation of $\alpha_s(M_Z^2)$, M_c and M_b Eqs. (5) have been used. μ has been varied between $\sqrt{s}/2$ and $2\sqrt{s}$.

experiment individually) to the theoretical predictions at the upper and lower endpoints as follows: Data below and up to 3.7 GeV are combined to determine the factor n_- which characterizes the mismatch between theory and experiment below threshold

$$n_- \equiv \left\langle \frac{R_{exp}(s)}{R_{pQCD}(s)} \right\rangle , \tag{7}$$

and an averaged experimental R value just below threshold

$$R_- \equiv n_- \, R_{pQCD}((3.7\text{GeV})^2) \, . \tag{8}$$

	PLUTO	DASP	MARK1
Interval below (GeV)	$3.6000 - 3.6600$	$3.6025 - 3.6500$	$3.0000 - 3.6500$
n_-	1.04 ± 0.01	1.06 ± 0.02	1.18 ± 0.04
R_-	2.25 ± 0.03	2.29 ± 0.04	2.55 ± 0.08
Interval above (GeV)	$4.9800 - 4.9800$	$5.0000 - 5.1950$	$5.1000 - 6.0000$
n_+	1.04 ± 0.01	1.15 ± 0.01	1.12 ± 0.01
R_+	3.85 ± 0.04	4.27 ± 0.04	4.14 ± 0.05
$\Delta\alpha^{(5)}_{c\bar{c}}(M_Z^2) \times 10^4$ (Model 1)	15.65 ± 0.19	15.26 ± 0.25	16.22 ± 0.32
$\Delta\alpha^{(5)}_{c\bar{c}}(M_Z^2) \times 10^4$ (Model 2)	15.64 ± 0.16	15.68 ± 0.30	15.83 ± 0.36

Table 2: Contribution to $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ from the energy interval 3.7 to 5.0 GeV.

In a similar way n_+ and R_+ are derived from the data around and above 5 GeV. The normalization factors and the combined R values as given in Tab. 2 are consistent with the systematic errors quoted by the experiments. To account for the difference between n_- and n_+ even within one experiment, two models are used for the interpolation into the interior of the interval. *Model 1:* The difference is due to the different efficiencies for final states with and without charmed mesons. *Model 2:* The difference is due to a linear s dependence of the experimental normalization and thus reflected in a linear s dependence of the quantity $1/n(s)$ in the threshold region.

The average of the two slightly different results is taken as central value and the three experiments are then assumed to be uncorrelated. The typical spread of ± 0.2 is taken as systematical uncertainty which is combined linearly with the statistical error for which we take the maximum of *Model 1* and *Model 2*. The combined result is thus given by

$$\Delta\alpha^{(5)}_{c\bar{c}}(M_Z^2) \quad = \quad (15.67 \pm 0.34) \times 10^{-4}\,. \tag{9}$$

A similar approach has been adopted in [5]. In [5], however, only MARK I data were employed (with DASP, PLUTO and Crystal Ball data used for cross checks), an energy independent correction factor was chosen, and the pQCD prediction for massless quarks was used for the calibration below charm threshold.

For the three lowest and narrow charmonium resonances we use the narrow width approximation:

$$\Delta\alpha^{(5)}_{R}(M_Z^2) \quad = \quad \frac{3}{\alpha}\left(\frac{\alpha}{\alpha(M_R^2)}\right)^2 \frac{M_Z^2}{M_R^2}\frac{M_R\Gamma_{ee}}{M_Z^2 - M_R^2}\,, \tag{10}$$

with $(\alpha/\alpha(M_\psi^2))^2 = 0.96$. M_R is the mass of the resonance and Γ_{ee} the partial width into electrons. The errors from the three charmonium resonances are added linearly. The result, given in Tab. 3, differs by about 0.7×10^{-4} from [4, 13].

Input	energy region	$\Delta\alpha^{(5)} \times 10^4$
low energy data [13]	$2m_\pi - 1.8$ GeV	56.90 ± 1.10
narrow charmonium resonances	$J/\Psi, \Psi(2S), \Psi(3770)$	9.24 ± 0.74
"normalized" data	$3.7 - 5.0$ GeV	15.67 ± 0.34
Υ resonances	$\Upsilon(1S) - \Upsilon(11.019)$	1.17 ± 0.09
interpolation of $b\bar{b}$	$11.075 - 11.2$	0.03 ± 0.03
pQCD (and QED)	$1.8 - \infty$	194.45 ± 0.96
	total	277.45 ± 1.68

Table 3: Contributions to $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ from different energy regions.

The contributions from the three lowest Υ resonances are evaluated through Eq. (10) with $(\alpha/\alpha(M_\Upsilon^2))^2 = 0.93$. For the bottom threshold an approach similar to the one for charm could be employed. However, the $b\bar{b}$ cross section between 10.5 and 11.075 GeV is saturated by the three Υ resonances at 10.580 GeV, 10.865 GeV and 11.019 GeV. (The result for the six Υ resonances as listed in Tab. 3 differs from [13] by 0.2×10^{-4}.) For energies above 11.2 GeV the perturbative treatment seems adequate. Between 11.075 and 11.2 GeV a linear increase from zero to the perturbative value is assumed, and the error is conservatively taken to be equal to this value.

For the low energy region up to 1.8 GeV we use the value $(56.90 \pm 1.10) \times 10^{-4}$ [13]. The individual results for the different regions are listed in Tab. 3. Combining the experimental errors, those from $\alpha_s^{(5)}(M_Z^2)$, the quark masses and the theoretical error in quadrature, we find

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) \;=\; (277.45 \pm 1.68) \times 10^{-4} \tag{11}$$

as our main result. Alternatively we could have used the more conservative analysis of [4] for the region up to 2.125 GeV with a contribution of $(63.42 \pm 2.59) \times 10^{-4}$ and pQCD only above 2.125 GeV. The result of this approach, $\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = (278.30 \pm 2.82) \times 10^{-4}$, would differ slightly in the central value and significantly in the size of the error.

Frequently the contribution from the top quark is added to this value. Using the three-loop QCD corrections [7]

$$\begin{aligned}\Delta\alpha_{\rm top}(s) \;=\; & -\frac{4}{45}\frac{\alpha}{\pi}\frac{s}{M_t^2}\left\{1 + 5.062\frac{\alpha_s^{(5)}(\mu^2)}{\pi} + \left(28.220 + 9.702\ln\frac{\mu^2}{M_t^2}\right)\left(\frac{\alpha_s^{(5)}(\mu^2)}{\pi}\right)^2\right. \\ & \left. + \frac{s}{M_t^2}\left[0.1071 + 0.8315\frac{\alpha_s^{(5)}(\mu^2)}{\pi} + \left(6.924 + 1.594\ln\frac{\mu^2}{M_t^2}\right)\left(\frac{\alpha_s^{(5)}(\mu^2)}{\pi}\right)^2\right]\right\},\end{aligned} \tag{12}$$

one obtains

$$\Delta\alpha_{\rm top}(M_Z^2) \;=\; (-0.70 \pm 0.05) \times 10^{-4}\,, \tag{13}$$

where we have used $M_t = 175.6 \pm 5.5$ GeV. For convenience Eq. (12) is expressed in terms of $\alpha_s^{(5)}(\mu^2)$. For the numerical evaluation $\alpha_s^{(5)}(M_Z^2) = 0.118$ has been chosen.

A major contribution to the vacuum polarization originates from the leptons. The dominant term is given by

$$\begin{aligned}\Delta\alpha_{\mathrm{lep}}(M_Z^2) &= \frac{\alpha}{\pi}\sum_{i\in\{e,\mu,\tau\}}\left(-\frac{5}{9}+\frac{1}{3}\ln\frac{M_Z^2}{m_i^2}-2\frac{m_i^2}{M_Z^2}+\mathcal{O}\left(\frac{m_i^4}{M_Z^4}\right)\right)\\ &\quad+\Delta\alpha_{\mathrm{lep,2l}}(M_Z^2)+\mathcal{O}\left(\alpha^3\right)\\ &\approx 314.19\times10^{-4}+\Delta\alpha_{\mathrm{lep,2l}}(M_Z^2)+\mathcal{O}\left(\alpha^3\right)\,. \end{aligned}\tag{14}$$

The two-loop correction [17]

$$\begin{aligned}\Delta\alpha_{\mathrm{lep,2l}}(M_Z^2) &= \left(\frac{\alpha}{\pi}\right)^2\sum_{i\in\{e,\mu,\tau\}}\left(-\frac{5}{24}+\zeta(3)+\frac{1}{4}\ln\frac{M_Z^2}{m_i^2}+3\frac{m_i^2}{M_Z^2}\ln\frac{M_Z^2}{m_i^2}+\mathcal{O}\left(\frac{m_i^4}{M_Z^4}\right)\right)\\ &\approx 0.78\times10^{-4}\end{aligned}\tag{15}$$

leads to a shift which could in principle become relevant in forthcoming precision studies.

For the combined result we thus obtain

$$\left(\alpha(M_Z^2)\right)^{-1} = 128.927 \pm 0.023, \tag{16}$$

if we use the more optimistic analysis [13] for the region below 1.8 GeV, and alternatively $\left(\alpha(M_Z^2)\right)^{-1} = 128.916 \pm 0.039$ if we employ the analysis from [4] below 2.125 GeV. In Tab. 4 our result for $\Delta\alpha_{\mathrm{had}}^{(5)}(M_Z^2)$ is compared to earlier evaluations. Our uncertainty is only a quarter of the one from the analysis of [1] or [4] based on data only — at the price of a more pronounced dependence on pQCD at relatively low energies. The reduction of the error by a factor 1.5 compared to [13] is to a large extend a consequence of our different treatment of the charm threshold. The shift of the central value by -1.0×10^{-4} compared to [13] is mainly due to different values for the charmonium and bottomonium contributions and our treatment of the charm threshold. In the prediction for $\alpha(M_Z^2)$ this is partly compensated by our inclusion of the leptonic two-loop contribution of 0.8×10^{-4}.

Summary: The effective fine structure constant at M_Z has been evaluated with input from pQCD over most of the integration region. A detailed analysis of the theoretical uncertainties has been performed. The two-loop piece for leptons has been included. In comparison with earlier results based on the analysis of data a slight shift of the central value and a significant reduction of the error has been obtained.

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$\Delta\alpha_{\rm had}^{(5)}(M_Z^2) \times 10^4$	Reference
273.2 ± 4.2	[5], Martin et al. '95
280 ± 7	[1], Eidelman et al. '95
280 ± 7	[2], Burkhardt et al. '95
275.2 ± 4.6	[3], Swartz '96
281.7 ± 6.2	[4], Alemany et al. '97
$278.4 \pm 2.6^*$	[13], Davier et al. '97
277.5 ± 1.7	this work

Table 4: Comparison of different evaluations of $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$. ($^*\Delta\alpha_{\rm top}(M_Z^2)$ subtracted.)

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