TTP98-01

January 1998

Coulomb resummation for $b\bar{b}$ system near threshold and precision determination of α_s and m_b.

J.H.Kühn,

Institut für Theoretische Teilchenphysik Universität Karlsruhe

7500 Karlsruhe 1, Germany

A.A.Penin and A.A.Pivovarov

Institute for Nuclear Research of the Russian Academy of Sciences,

60th October Anniversary Pr., 7a, Moscow 117312, Russia

Abstract

We analyze sum rules for the Υ system with resummation of threshold effects on the basis of the nonrelativistic Coulomb approximation. We found the pole mass of the bottom quark $m_b = 4.75 \pm 0.04$ GeV and the strong coupling constant $\alpha_s(M_Z) = 0.118 \pm 0.006$. The origin of the contradiction between two recent estimates obtained within the same formal framework is clarified.

1 Introduction

The knowledge of precise numerical values of the coupling constant and quark masses is important for obtaining accurate predictions within QCD. The technique of sum rules is extensively used both for the determination of these parameters from experimental data on low lying resonances and for calculation of properties of higher excited states [1, 2]. The Υ system is a rather clean place to obtain the values of the strong coupling constant α_s and also of the b-quark pole mass m_b that is a key parameter for rich B-meson physics. In sum rules for the Υ system there are some subtleties that make them interesting by themselves, the main one being the importance of Coulomb effects and the applicability of the nonrelativistic approximation as a rather good starting point for the expansion in the coupling constant α_s and b-quark velocity v that both are small parameters of the problem [3]. The ordinary perturbation theory (PT) expressions for several first moments of the spectral density are now available with α_s^2 accuracy [4] but they cannot be used in theoretical formulas for sum rules directly because the spectrum is well known experimentally only for energies close to threshold due to existence of sharp resonances while the contribution of the continuum to these low moments is large in comparison with the resonance contribution. For higher moments that are saturated by these low energy resonances with better accuracy the ordinary PT expansion in the coupling constant is not applicable (coefficients of the expansion grow fast with the order of the moment) and a kind of resummation is required. In this paper we analyze the sum rules for the Υ system with resummation of Coulomb effects that are largely responsible for the growth of PT coefficients to extract the values of α_s and m_b. We consistently take into account the first order corrections in α_s and give an estimate of the magnitude of the high order corrections.

The analysis based on the same formal approach has been already performed in

[5, 6] along different lines and with contradictory results. In ref. [5] next-to-leading approximation in α_s both to the high energy part of the correlator and to the Coulomb potential was used while in ref. [6] further terms of perturbative expansion in α_s have been partly included as well. This allows one no direct comparison of the results of those two papers though in each case there is a claim that corrections are small and under control. However the numerical results do not overlap within given error bars and a look into the origin of this difference is in order. We find that the difference is essentially because in ref. [6] the analysis in the scheme with the pole mass is parametrically different from that of ref. [5] – the contribution of Coulomb poles to the theoretical moments is missed. We demonstrate that after a restoration of the contribution of the Coulomb poles the results of two papers can be made completely compatible by the corresponding choice of the scale in the nonrelativistic Coulomb potential – the low one ($\mu \sim 1$ GeV) for the case of ref. [5] and of order of the b-quark mass for ref. [6].

Then the result of ref. [5] for the value of $\alpha_s(M_Z)$ is in a contradiction with the LEP data [7]. We find that the main uncertainty enters the theoretical predictions through the high order α_s corrections which become uncontrollable at the low normalization point used in ref. [5]. So the uncertainties of the predictions given in ref. [5] are strongly underestimated that is the reason of the discrepancy with the LEP value for $\alpha_s(M_Z)$.

The paper is organized as follows. The stage is set in Sect. 2 which is essentially for fixing our notations. The numerical analysis of sum rules is done in Sect. 3 where the procedure of optimization for the nonrelativistic Coulomb Green function is described. In Sect. 4 we discuss the difference between the present analysis and those of refs. [5, 6]. Sect. 5 contains our conclusions. Some explicit formulas are given in the Appendix.

2 Sum rules and nonrelativistic approximation

We perform our analysis directly computing moments on the theoretical side of the sum rules instead of using a representation for them through the propagation function of nonrelativistic approximation in Euclidean time as has been done in ref. [5]. The difference between these two approaches is briefly discussed later.

The moments of the spectral density are defined to be dimensionless and are given by the following expression

$$\mathcal{M}_n = \frac{12\pi^2}{n!}(4m_b^2)^n \frac{d^n}{ds^n}\Pi(s)\bigg|_{s=0} = (4m_b^2)^n \int_0^\infty \frac{R(s)ds}{s^{n+1}}$$

where

$$R(s) = 12\pi \mathrm{Im}\Pi(s+i\epsilon)$$

and $\Pi(s)$ is a polarization function of electromagnetic current of b-quark

$$\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2) = i\int dx e^{iqx}\langle 0|Tj_\mu(x)j_\nu(0)|0\rangle,$$

$j_\mu = \bar{b}\gamma_\mu b$. Because of the quark-hadron duality assumption the imaginary part of the polarization function $R(s)$ is related to the experimentally measurable quantity $R_b(s) = \sigma(e^+e^- \to \mathrm{hadrons}_{b\bar{b}})/\sigma(e^+e^- \to \mu^+\mu^-)$ by the sum rules

$$\mathcal{M}_n = (4m_b^2)^n \int_0^\infty \frac{R(s)ds}{s^{n+1}} = \frac{(4m_b^2)^n}{Q_b^2}\int_0^\infty \frac{R_b(s)ds}{s^{n+1}}$$

with $Q_b = -1/3$ being the electric charge of the b-quark.

The experimental values of the moments are obtained by using information on the first six Υ resonances. Their leptonic widths Γ_k and masses M_k, $(k = 1\ldots 6)$ are known rather precisely [7]

$$\mathcal{M}_n = (4m_b^2)^n \left(\frac{9\pi}{\alpha_{QED}^2(m_b)Q_b^2}\sum_{k=1}^{6}\frac{\Gamma_k}{M_k^{2n+1}} + \int_{s_0}^\infty \mathrm{d}s\frac{R_b(s)}{s^{n+1}}\right) \qquad (1)$$

while the last term is a continuum contribution. Beyond the resonance region, for energies larger than $s_0 \approx (11.2\ \mathrm{GeV})^2$, the spectral density $R_b(s)$ can be approximated by the ordinary PT expression for the theoretical spectral density $R(s)$. The influence of continuum on high moments is almost negligible numerically and in any case under strict control. Here $\alpha^2_{QED}(m_b) = 1.07\alpha^2$ [5]. The theoretical expression of the polarization function in the nonrelativistic approximation that is justified for the energy region near the threshold is given by [8]

$$\Pi(s) = \left(1 - 4C_F\frac{\alpha_s}{\pi}\right)\frac{3}{2m_b^2}G(0,0,k) \equiv N_{pt}\frac{3}{2m_b^2}G(0,0,k) \tag{2}$$

where $C_F = 4/3$. The energy of the nonrelativistic Coulomb $b\bar{b}$ system counted from the threshold is $E = -k^2/m_b$, and there are several ways of introducing k through the relativistic parameter of energy square s

$$k = \begin{cases} \frac{1}{2}\sqrt{4m_b^2 - s} \\ m_b\sqrt{-1 + 4m_b^2/s} \\ \sqrt{m_b(-\sqrt{s} + 2m_b)} \end{cases} \tag{3}$$

which all are equivalent in nonrelativistic approximation near the threshold. For $E < 0$ ($s < 4m_b^2$) the parameter k is real while for $E > 0$ ($s > 4m_b^2$) the parameter k is purely imaginary. The quantity $G(0,0,E)$ is the nonrelativistic Green function (GF) of the $b\bar{b}$ system that can be found exactly for Coulomb potential which is supposed to dominate the whole QCD interaction in the energy region of interest. The $\alpha_s(\mu)$ correction in eq. (2) (so called perturnative factor N_{pt}) reflects the contribution of high energies to the polarization function and has a natural scale for the perturbation theory of the order of m_b, $\mu \sim m_b$. We use $\overline{\mathrm{MS}}$ scheme of renormalization throughout the paper and m_b is a pole mass of the b quark as has been already mentioned in the Introduction. The PT expression for the nonrelativistic potential is known up to the second order in α_s [9]

$$V = V_C + \Delta V = V_C + \frac{\alpha_s}{4\pi}V_1 + \left(\frac{\alpha_s}{4\pi}\right)^2 V_2 \tag{4}$$

where V_C is a Coulomb potential, $V_C = -C_F\alpha_s/r$ and explicit expressions for $V_{1,2}$ are given in the Appendix (eq. (16)). The corrections to the Coulomb potential in eq. (4) generate corrections to the Coulomb GF G_C. The full nonrelativistic GF in condensed notation (see Appendix for more details) is given by q

$$G = G_C + \Delta G = G_C - G_C \otimes \Delta V \otimes G_C = G_C + G_1 + G_2. \tag{5}$$

where G_C is a Coulomb GF and G_1 and G_2 correspond to V_1 and V_2 contributions (see eqs. (18, 20, 21) of Appendix).

In principle one can develop a regular expansion for the polarization operator near the threshold using Bethe-Salpeter equation for four-point amplitude using Barbiery-Remiddy exact solution [10] as a starting point. This is however a rather difficult technical problem. Meanwhile up to the first order in α_s eqs. (2, 5) give a complete result. The self-consistent calculation of the expansion of the polarization function beyond the first order is not available, so we will use the G_2 term of eq. (5) not for the regular computation but rather for estimating the typical high order α_s contributions to the sum rules.

3 Numerical analysis of sum rules

We consider two methods to compute the nonrelativistic GF when higher order perturbative corrections to the potential are included. The first one is the direct computation when the splitting into the leading term (Coulomb) and corrections is determined by the ordering in the $\overline{\text{MS}}$ scheme, i.e. we count powers of α_s. This method in fact fails to deal with corrections keeping them under control. The second methods is the optimized computation when the splitting of the whole potential into the leading term (Coulomb) and corrections is determined by the physics of the problem rather than the artificial $\overline{\text{MS}}$ charge. This technique gives solid results when all expansions (for the effective charge and for the nonrelativistic GF) converge well for the moments that are computed.

3.1 Direct computation

To compute the theoretical part of sum rules we use the approximation (2) for the polarization function. In terms of the nonrelativistic variable of energy E, the quantity $G(0,0,E)$, the exact nonrelativistic Coulomb Green function of the $b\bar{b}$ system, has the poles below threshold at real $E < 0$ or $s < 4m_b^2$ corresponding to the Coulomb bound states and a cut above the threshold at real $E > 0$ ($s > 4m_b^2$). Accordingly, the theoretical moments are split up into the following sum

$$\mathcal{M}_n = \mathcal{P}_n + \mathcal{C}_n$$

where $\mathcal{P}_n$ is the pole contribution from the energy region below the threshold

$$\mathcal{P}_n = (4m_b^2)^n \int_0^{4m_b^2} \frac{R(s)ds}{s^{n+1}}$$

and $\mathcal{C}_n$ is the continuum contribution from the energy region above the threshold

$$\mathcal{C}_n = (4m_b^2)^n \int_{4m_b^2}^{\infty} \frac{R(s)ds}{s^{n+1}}.$$

n	$\mathcal{P}_n^C$	$\mathcal{P}_n^1$	$\mathcal{P}_n^2$
10	0.16017	0.23375	0.25756
12	0.16631	0.25180	0.27743
14	0.17271	0.27101	0.29859
16	0.17939	0.29145	0.32111
18	0.18635	0.31319	0.34507
20	0.19360	0.33630	0.37057

Table 1: The 0th, 1st and 2nd order pole contributions to the moments

Then the total moments are represented by the sum

$$\mathcal{M}_n = \mathcal{M}_n^C + \mathcal{M}_n^1 + \mathcal{M}_n^2 \tag{6}$$

where $\mathcal{M}_n^C$ is the contribution of a pure Coulomb part, $\mathcal{M}_n^i$ are the contributions due to G_i terms in eq. (5). The same representation is used for $\mathcal{P}_n$ and $\mathcal{C}_n$ parts separately and also for $\Pi(s)$ and $R(s)$, for instance, $\mathcal{P}_n = \mathcal{P}_n^C + \mathcal{P}_n^1 + \mathcal{P}_n^2$ and so on. An example of the breakdown of the total numerical values of the moments into particular contributions due to G_i terms for $\mu = m_b = 4.75$ GeV, $\alpha_s(M_Z) = 0.118$, that are chosen as our typical values (see below), is given in Tables 1, 2.

n	$\mathcal{C}_n^C$	$\mathcal{C}_n^1$	$\mathcal{C}_n^2$
10	0.26388	-0.02508	-0.00997
12	0.22195	-0.02055	-0.00678
14	0.19059	-0.01724	-0.00440
16	0.16652	-0.01477	-0.00259
18	0.14763	-0.01288	-0.00119
20	0.13247	-0.01139	-0.00009

Table 2: The 0th, 1st and 2nd order continuum contributions to the moments

From the Tables 1,2 one sees that the expansion (5) does not work for the pole contribution even at relatively large scale ($\mu = m_b$), i.e. next terms are larger than the lower order terms. In Voloshin's paper this kind of corrections was treated by redefinition of the scale in the leading order approximation that resulted in introduction of the function $h(\gamma)$ that determines the appropriate scale for a given moment. Because only the first order corrections (V_1 from Eq. (4)) have been taken into account (all contributions due to V_2 were nullified) such a redefinition of the scale in the leading order approximation does allow to sum them completely into the Coulomb Green function, i.e. by requiring $G_1 = 0$. However, this leads to the extremely low normalization point where the ordinary PT for the potential breaks down and high order corrections become uncontrollable.

3.2 Optimized computation

Now we turn to our approach of computation of nonrelativistic Green function with optimization of contributions due to higher order corrections to the potential. In contrast to QED where the Coulomb term exhausts the potential in the nonrelativistic approximation and hence generates the full Green function, in QCD the potential itself is a PT expansion in α_s. Due to the presence of $\log(\mu r)$ terms in this expansion the nonrelativistic Green function cannot be obtained in the closed form and one has to use some approximation. The perturbative expansion in the corrections to the potential looks natural for this purpose. Now, for computation of the best approximation for the nonrelativistic GF (5) we apply the technique of optimization of the expansion (5) with regard to higher order corrections using the freedom of redefinition of the leading order approximation by the proper choice of the Coulomb coupling constant [11]. It is clear that the α_s in the $\overline{\mathrm{MS}}$ scheme is not necessarily the best parameter for the Coulomb problem as a leading approximation for the nonrelativistic GF and the choice of the more suitable charge to substitute into the expression for the Coulomb GF will give better results. Thus we introduce an effective Coulomb charge by adding a part of the higher order constant contribution to the leading term. We parametrize this part with a parameter t. To a great extent this remind the simple change of the scale μ though in reality it is a more sophisticated procedure. Its real content is to minimize (or optimize) the contribution of the logs terms in the potential that we cannot treat exactly. The μ dependence is still controlled by the ordinary renormalization group invariance. Moreover, in contrast to ref. [5] where different normalization points are used for the analysis of the soft and hard corrections, in our approach we may use a unique expansion parameter $\alpha_s(m_b)$ and the corrections related to the matching are absent. Thus, the Coulomb part of the GF is determined not by the $\alpha_s(\mu)$ defined in the $\overline{\mathrm{MS}}$ scheme but an effective Coulomb parameter $\alpha_C(t)$

$$\tilde{V}_C = -\frac{C_F \alpha_C(t)}{r} \tag{7}$$

with $\alpha_C(t)$ being a function of α_s

$$\alpha_C(t) = \alpha_s \left(1 + \frac{\alpha_s}{4\pi}(1-t)C_0^1\right). \tag{8}$$

We give an explicit formula for such a decomposition to one loop (first order approximation for the potential)

$$\begin{aligned} V(r) &= -\frac{C_F \alpha_s}{r}\left(1 + \frac{\alpha_s}{4\pi}(C_0^1 + C_1^1 \ln(r\mu))\right) \\ &= -\frac{C_F \alpha_s}{r}\left(1 + \frac{\alpha_s}{4\pi}(1-t)C_0^1\right) - \frac{C_F \alpha_s}{r}\frac{\alpha_s}{4\pi}\left(tC_0^1 + C_1^1 \ln(r\mu)\right) \\ &\equiv \tilde{V}_C + \frac{\alpha_s}{4\pi}\tilde{V}_1. \end{aligned} \tag{9}$$

Now

$$\tilde{V}_C + \frac{\alpha_s}{4\pi}\tilde{V}_1 = V_C + \frac{\alpha_s}{4\pi}V_1$$

where the effective Coulomb term $\tilde{V}_C$ is given by (7) with the effective charge (8) and $\tilde{V}_1$ is considered as a correction. Thus we have added a part of the constant of the second order term (V_1) to the Coulomb charge of the leading approximation. Developing the expansion around Coulomb GF corresponding to $\tilde{V}_C$ we minimize corrections by choosing the proper numerical value of the parameter t. At the final stage we have to check that both expansions (for the nonrelativistic GF around the Coulomb approximation and for the effective charge around the α_s) converge well. This happens to be the case for the considered system. If it wasn't so one would have to conclude that the problem cannot be treated perturbatively, i.e. one cannot use the PT expression for the potential to generate the nonrelativistic approximation. Technically the method results in substituting the combination $C_F\alpha_s$ by $C_F\alpha_C$ in G_C

and G_1. Then we choose $t(n)$ (or α_C) so that $\mathcal{M}_n^1 = 0$ to get the best convergence of the series (6).

In the analysis we use moments with $10 < n < 20$. For this range of moments the contribution of the continuum (which is not well known experimentally) is sufficiently suppressed for the results being practically independent of s_0 and, on the other hand, the nonperturbative power corrections due to the gluonic condensate are also suppressed [5]. We use the normalization point $\mu = m_b$ and take into account the first order α_s corrections to the polarization function (2).

The result of the fit for α_s is $\alpha_s(m_b) = 0.22 \pm 0.02$ or $\alpha_s(M_Z) = 0.118 \pm 0.006$ that is in a good agreement with other estimates (see ref. [12] as a review). The sum rules are much more sensitive to the b-quark mass than to the strong coupling constant so it is instructive to fix $\alpha_s(M_Z) = 0.118$ that is the "world average" value and then to extract $m_b(n)$. For the fixed $\alpha_s(M_Z) = 0.118$ the $m_b(n)$ distribution as a function of n is given in Table 3.

n	m_b(GeV)	$\mathcal{P}_n$	$\mathcal{C}_n$	α_C/α_s
10	4.730	0.4143	0.3308	1.306
12	4.742	0.4723	0.2827	1.328
14	4.752	0.5391	0.2481	1.349
16	4.760	0.6028	0.2174	1.362
18	4.767	0.6786	0.1949	1.376
20	4.774	0.7642	0.1780	1.389

Table 3: The n distribution of m_b, $\mathcal{P}_n$, $\mathcal{C}_n$ and α_C/α_s for $\mu = m_b$, $\alpha_s(M_Z) = 0.118$

Our final estimate of the b-quark mass is $m_b = 4.75$ GeV which gives the average over the range $n = 10 - 20$ (second column in Table (3)). As we see, the pole contribution exceeds the continuum one and obviously cannot be neglected as it was done in ref. [6]. What is the uncertainty of this result? The total error coming from the specific choice of $k(s)$ function in (3) and of numerical value for s_0 is about $\pm 0.1\%$. However, the main uncertainty comes from higher orders of PT for the potential (4) or from the expansion of the GF in eq. (5). One can estimate the higher order corrections from n distribution (Table 3) or from μ dependence of the results which formally must be canceled by the high order contributions (Table 4).

μ(GeV)	m_b(GeV)	N_{pt}	α_C/α_s	$\alpha_s(\mu)$
m_b	4.752	0.631	1.349	0.218
4.0	4.763	0.608	1.254	0.231
3.0	4.782	0.564	1.245	0.257
2.0	4.799	0.478	1.118	0.308
1.2	4.767	0.273	0.879	0.428

Table 4: The μ dependence of m_b, N_{pt}, α_C/α_s and α_s for $n = 14$, $\alpha_s(M_Z) = 0.118$

Note that at the scale $\mu \sim m_b$ both the hard gluon corrections which are parametrized by the factor

$$N_{pt} = 1 - 4C_F \frac{\alpha_s}{\pi} \tag{10}$$

and the soft gluon corrections parametrized by the ratio α_C/α_s are of the same order $\sim 35\%$ so this seems to be an optimal normalization point. For lower scale the hard corrections become large while for higher scale the same is true for the soft corrections. These corrections can be also estimated directly by adding G_2 part of the nonrelativistic GF. The relative weight of α_s corrections can be found from Table 5. In the Coulomb approximation the hard gluon corrections, *i.e.* the perturbative factor in eq. (2) is neglected while the 1st order corrections include this factor and G_1 part of the nonrelativistic GF. In the second order estimate we take into account G_2 term but do not use the optimization procedure. Thus collecting all the data we estimate the high order corrections to the b-quark mass to be about 1%. Our final value is $m_b = 4.75 \pm 0.04$ GeV.

Order of approximation	m_b(GeV)
pure Coulomb approximation	4.700
total 1st order α_s corrections included	4.752
2nd order α_s corrections to the potential included	4.776

Table 5: The convergence of the perturbation theory for m_b at $n = 14$, $\mu = m_b$, $\alpha_s(M_Z) = 0.118$

Stress again, that we have used the α_s^2 corrections to the potential (4) only as an estimate of the typical size of all α_s^2 corrections which is a pure assumption. The final answer in the next-to-leading order can be obtained only after full computation of α_s^2 corrections (including relativistic ones) to the whole approximation for the polarization function.

4 Comparison with the previous analyses

In Voloshin's paper [5] the low normalization point $\mu = 1$ GeV is used which is effectively the way of reshuffling the PT series in higher orders. The hard gluon correction was separated and taken at $\mu = 0.632 m_b$. Using these prescriptions we reproduce the results of ref. [5] $m_b = 4.827 \pm 0.007$ GeV, $\alpha_s(M_Z) = 0.109 \pm 0.001$ within our approach.

In our opinion, the presicion of these results is however strongly overestimated: no variation due to higher order corrections (which at this level reduces effectively to the change of the parameter μ) is allowed. Given errors just reflect the uncertainties of estimates stemming from different moments while the procedure of treating μ dependence is fixed. Also, though the correction induced by using the different normalization points for hard and soft region is formally of the next (second) order in α_s they can be very essential because of the low normalization point. This can be clearly seen from the fact that at $\mu \sim 1$ GeV the first order hard gluon corrections are about 80% so that PT does not converge. On the other hand we found that the second order correction to the moments due to V_2 term of the potential (4) exceeds 100% at $\mu \sim 1$ GeV so the first order PT can be hardly used for the analysis of the soft gluon corrections at this scale though the first order corrections are minimized (equal to zero). The error bars given in ref. [5] do not account for high order corrections. This is the reason for the discrepancy between the result of ref. [5] and LEP data on $\alpha_s(M_Z)$ [7].

Speaking of uncertainties we would like to touch the point of using the moments in relativistic representation against the Laplace transform of the nonrelativistic GF. We mentioned this problem in the beginning of Sect. 2 when explaining details of our treatment of nonrelativistic approximation. Corrections of order $1/n$ are different in these approaches. Even more, the parametric estimate of $O(1/n)$ for difference between moments and Laplace transform in ref. [5] is incorrect. Let us demonstrate it in more detail now. Two formulas for the moments

$$(4m^2)^n \int \frac{R(s)ds}{s^{n+1}} \qquad \text{vs} \qquad \int R(E)e^{-\frac{E}{m}n}\frac{dE}{m}$$

that are used and where $s=(2m+E)^2$ are not equivalent up to $1/n$ terms for large n for any function $R(s)$ but only for rather smooth ones. In fact, the dominant contribution to the theoretical value of the moments is due to Coulomb bound states which lead to δ-function singularities in $R(s)$. So, one has to check the equality

$$(4m_b^2)^n \int \frac{R(s)ds}{s^{n+1}} \overset{?}{\doteq} \int R(E)e^{-\frac{E}{m_b}n}\frac{dE}{m_b}(1+O(\frac{1}{n})) \tag{11}$$

for $R(s)=4m_b^2\delta(s-m_C^2)$ with $m_C=2m_b(1+\Delta)$ where $\Delta\sim\alpha_s^2$, $\Delta<0$. Then one has

$$(4m_b^2)^n\int_0^\infty \frac{R(s)ds}{s^{n+1}}=\frac{(4m_b^2)^{n+1}}{(m_C^2)^{n+1}}=\frac{1}{(1+\Delta)^{2n+2}}$$

while the right hand side of eq. (11) is

$$\int_0^\infty e^{-\frac{E}{m_b}n}R(E)\frac{dE}{m_b}=\frac{e^{-2\Delta n}}{1+\Delta}$$

or

$$\frac{1}{(1+\Delta)^{2n+1}}\qquad \text{vs}\qquad e^{-2\Delta n} \tag{12}$$

and these two quantities are not related by the estimate (11), i.e. with an accuracy of $O(1/n)$ uniformly in n and Δ. Indeed for large n and fixed negative Δ one has

$$\left.\frac{e^{2\Delta n}}{(1+\Delta)^{2n+1}}\right|_{n\to\infty}=0\neq 1+O\left(\frac{1}{n}\right).$$

They are close to each other with the stated accuracy only for $|\Delta|n<<1$ that can be easily checked by expanding both quantities in a Taylor series for small $|\Delta|$. For a smooth function, however, the estimate (11) is valid. For instance, for $R(s)=(s/4m_b^2)\theta(s-4m_b^2)=(1+E/2m_b)^2\theta(E)$ one has

$$(4m_b^2)^n\int_{4m_b^2}^\infty \frac{R(s)ds}{s^{n+1}}=(4m_b^2)^{n-1}\int_{4m_b^2}^\infty\frac{sds}{s^{n+1}}=\frac{1}{n-1}=\frac{1}{n}+\frac{1}{n^2}+\ldots \tag{13}$$

while

$$\int_0^\infty e^{-\frac{E}{m_b}n}R(E)\frac{dE}{m_b}=\int_0^\infty e^{-\frac{E}{m_b}n}\left(1+\frac{E}{m_b}+\frac{E^2}{4m_b^2}\right)\frac{dE}{m_b}=\frac{1}{n}+\frac{1}{n^2}+\frac{1}{2n^3} \tag{14}$$

in accordance with eq. (13) and eq. (11). This observation in fact gives an additional constraint from above on possible n. Just due to this mathematical remark the nonrelativistic propagator of the Coulomb system in Euclidean time used in ref. [5] does not approximate the experimental moments computed from the standard definition (1) with the stated parametric accuracy. This introduces numerical uncertainty which has not been taken into account before.

The results of ref. [6] $m_b = 4.60\pm0.02$ GeV, $\alpha_s(M_Z) = 0.119\pm0.008$ are reproduced as soon as the pole term is omitted. This is the reason for having smaller value of m_b. In fact, omitting the pole contribution leads to a huge (about 50%) change of the moments (see Table 3) but only to $\sim 3\%$ variation of the b-quark mass. Then the way the potential is treated in ref. [6] seems not to be correct to us. A simple use of the running coupling constant $\alpha_V(q)$ with $q = \sqrt{s - 4m_b^2}$ which is defined in all orders of PT as $V(r) = -C_F\alpha_V(r)/r$ instead of α_s in the leading order nonrelativistic result (Sommerfeld factor) [6] is not consistent with the expansion of the GF (5) that is generated by PT of the potential (4) though it does catch a part of the contribution. This can be directly found by comparing the data listed in Table 2 and Table 6 which contains the continuum contributions $\mathcal{Z}_n$ to the moments computed according to the prescription of ref. [6] by inserting the running coupling $\alpha_V(q)$ into the Sommerfeld factor. Note that in our approach this procedure corresponds to inserting $\alpha_V(q)$ instead of α_s to the Coulomb GF G_C. In Table 6 the quantities $\mathcal{Z}_n^i$ are defined as follows

$$\mathcal{Z}_n^i = (4m_b^2)^n \int_{4m_b^2}^{\infty} \frac{ds}{s^{n+1}} R^C(s)|_{\alpha_s \to \alpha_V^{(i)}(q)} \tag{15}$$

where α_V^i is a series for α_V in α_s up to the ith order

$$\alpha_V^{(0)} = \alpha_s, \qquad \alpha_V^{(1)}(q) = \alpha_s\left(1 + \frac{\alpha_s}{4\pi}\left(a_1 - 2\beta_0 \ln\left(\frac{q}{\mu}\right)\right)\right), \qquad \dots$$

($\alpha_V^{(2)}(q)$ and numerical values of parameters can be directly found from eqs. (16, 17) in the Appendix). The quantity $R^C(s)$ for $s > 4m_b^2$ reads

$$R^C(s) = \frac{18\pi}{m_b^2} N_{pt}\, \mathrm{Im} G_C(s + i\epsilon) = \frac{9}{2} N_{pt} \left(\frac{\pi C_F \alpha_s}{1 - \exp(-\pi C_F \alpha_s / v)}\right),$$

v being the b-quark velocity. Note that in eq. (15) α_s is not substituted by α_V in the perturbative factor N_{pt} (2,10). Thus the quantities $\mathcal{Z}_n^1$ and $\mathcal{Z}_n^2$ should be compared with $\mathcal{C}_n^C + \mathcal{C}_n^1$ and $\mathcal{C}_n^C + \mathcal{C}_n^1 + \mathcal{C}_n^2$ respectively. From Tables 2, 6 one finds that within the approach of ref. [6] the corrections to the leading order continuum contribution are about 2-4 times underestimated. However the main difference between the present analysis and that of ref. [6] is just an account for the pole contribution.

n	$\mathcal{Z}_n^0 \equiv \mathcal{C}_n^C$	$\mathcal{Z}_n^1$	$\mathcal{Z}_n^2$
10	0.26388	0.25534	0.25273
12	0.22195	0.21564	0.21380
14	0.19059	0.18582	0.18450
16	0.16652	0.16285	0.16188
18	0.14763	0.14475	0.14403
20	0.13247	0.13018	0.12964

Table 6: The corrections to the continuum contribution computed by inserting the running coupling $\alpha_V(k^2)$ into the Sommerfeld factor for $\mu = m_b$, $\alpha_s(M_Z) = 0.118$

Then in ref. [6] the results have much weaker μ dependence and the contribution of high orders of PT is claimed to be much smaller. The reason is that the main corrections and the strong μ dependence come from the pole contribution which is omitted in ref. [6].

5 Conclusion.

We reanalyzed the sum rules for Υ system and obtained new numerical estimates of the strong coupling constant and the bottom quark pole mass. Our results are different and lie outside error bars given by the authors of two existing papers on this subject, one of which uses first order analysis with Laplace transform of nonrelativistic GF taken in Coulomb approximation [5] and the other uses direct moments and takes into account the second order corrections in α_s [6]. Our results and the reasons for the discrepancy are:

- We found $m_b = 4.75 \pm 0.04$ GeV and $\alpha_s(M_Z) = 0.118 \pm 0.006$ which is consistent with the LEP data. The results are insensitive to the choice of k in eq. (3), s_0, and whether resumed or plain PT is used for integration beyond s_0. Main theoretical uncertainty of these values comes from the high order α_s corrections which induce n and μ dependence of the result. It is explicitly shown that the part of them connected to the potential is large that leads to less accuracy in estimate of the mass and the coupling constant than it was claimed before. Thus the main problem at the present stage of calculation is the consistent treatment of the next-to-leading α_s corrections.

- In the paper [6] the pole contribution to the theoretical spectral density is omitted. This results in the large difference in determination of the absolute value of m_b (which we consider as erroneous one) and gives the impressive stability against variation of the scale μ (or against inclusion of higher orders of PT) because it is the pole part that is strongly μ dependent.

- In the paper [5] the low normalization scale is used where the validity of the expansion around Coulomb solution due to huge high orders α_s corrections is quastionable. The uncertainty of results of ref. [5] is strongly underestimated.

Acknowledgements

A.A.Pivovarov acknowledges discussions with K.G.Chetyrkin and K.Melnikov. This work is partially supported by BMBF under contract No. 057KA92P and by Volkswagen Foundation under contract No. I/73611. A.A.Pivovarov is supported in part by the Russian Fund for Basic Research under contracts Nos. 96-01-01860 and 97-02-17065. A.A.Penin greatfully acknowledges partial financial support by the Russian Fund for Basic Research under contract 97-02-17065.

References

[1] V.A.Novikov *et al.*, Phys.Rev.Lett. **38**(1977)626;
V.A.Novikov *et al.*, Phys.Rep. **C41**(1978)1.

[2] N.V.Krasnikov and A.A.Pivovarov, Phys.Lett. **112B**(1982)397.

[3] M.B.Voloshin, Yad.Fiz. **36**(1982)247;
M.B.Voloshin and Yu.M.Zaitsev, Usp.Fiz.Nauk **152**(1987)361.

[4] K.G.Chetyrkin, J.H.Kühn, M.Steinhauser, Nucl.Phys. **B482**(1996)213.

[5] M.Voloshin, Int.J.Mod.Phys. **A10**(1995)2865.

[6] M.Jamin, A.Pich, Nucl.Phys. **B507**(1997)334.

[7] Particle Data Groop, Phys.Rev. **D54**(1996)1.

[8] M.A.Braun, ZhETP Lett. **27**(1968)652;
R.Barbieri, P.Christillin and E.Remiddi, Phys.Rev. **A8**(1973)2266.

[9] M.Peter, Phys.Rev.Lett. **78**(1997)602.

[10] E.E.Salpeter and H.A.Bethe, Phys.Rev. **84**(1951)1232;
R.Barbieri and E.Remiddi, Nucl.Phys. **B141**(1978)413.

[11] A.A.Penin and A.A.Pivovarov, Phys.Lett. **B367**(1996)342.

[12] G. Altarelli, Preprint **CERN-TH-96-265**, hep-ph/9611239.

Appendix

The corrections to the Coulomb potential read [9]

$$V_C(r) = -\frac{C_F\alpha_s(\mu)}{r},$$

$$V_1(r) = V_C(r)(C_0^1 + C_1^1 \ln(r\mu)), \tag{16}$$

$$V_2(r) = V_C(r)(C_0^2 + C_1^2 \ln(r\mu) + C_2^2 \ln^2(r\mu)),$$

where

$$C_0^1 = a_1 + 2\beta_0\gamma_E,$$

$$C_0^2 = \left(\frac{\pi^2}{3} + \gamma_E^2\right)\beta_0^2 + 2(\beta_1 + 2\beta_0 a_1)\gamma_E + a_2,$$

$$C_1^1 = 2\beta_0,$$

$$C_1^2 = 2(\beta_1 + 2\beta_0 a_1) + 8\beta_0^2\gamma_E,$$

$$C_2^2 = 4\beta_0^2, \tag{17}$$

$$a_1 = \frac{31}{9}C_A - \frac{20}{9}T_F n_f,$$

$$a_2 = \left(\frac{4343}{162} + 6\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right)C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3)\right)C_A T_F n_f -$$

$$-\left(\frac{55}{3} + 16\zeta(3)\right)C_F T_F n_f + \left(\frac{20}{9}T_F n_f\right),$$

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f, \qquad \beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A T_F n_f - 4C_F T_F n_f,$$

$$C_A = 3, \qquad C_F = \frac{4}{3}, \qquad T_F = \frac{1}{2},$$

$\gamma_E = 0.57\ldots$ is the Euler constant, $\zeta(x)$ is the Riemann ζ-function, and n_f is the number of light flavors.

The nonrelativistic Coulomb GF reads [8]

$$G_C(0,0,k) = -\frac{C_F\alpha_s m_b^2}{4\pi}\left(\frac{4k}{C_F\alpha_s m_b} + \ln\left(\frac{2k}{C_F\alpha_s m_b}\right) - \sum_{m=0}^{\infty} F(m)\right) \tag{18}$$

where

$$F(m) = \frac{1}{m + 1 - \dfrac{C_F\alpha_s m_b}{2k}} - \frac{1}{m+1}. \tag{19}$$

Evaluation of the corrections to the Coulomb GF is straitforward but rather complicated. In the calculation we used the representation of the Coulomb GF $G_C(x,y,k)$ as a sum of Laguerre polynomials [3]. The result reads

$$G_1(0,0,k) = \frac{\alpha_s}{4\pi}\frac{C_F\alpha_s m_b^2}{4\pi}\left(\sum_{m=0}^{\infty} F^2(m)(m+1)\left(C_0^1 - \ln\left(\frac{2k}{\mu}\right)C_1^1 + \Psi_1(m+2)C_1^1\right) - \right.$$
$$2\sum_{m=1}^{\infty}\sum_{n=0}^{m-1} F(m)F(n)\frac{n+1}{m-n}C_1^1 +$$
$$2\sum_{m=0}^{\infty} F(m)\left(\left(C_0^1 - \ln\left(\frac{2k}{\mu}\right)C_1^1\right) - (2\gamma_E + \Psi_1(m+1))C_1^1\right) + \tag{20}$$
$$\left. 2\left(\gamma_E\ln\left(\frac{2k}{\mu}\right) + \frac{1}{2}\ln^2\left(\frac{2k}{\mu}\right)\right)C_1^1\right),$$

$$G_2(0,0,k) = \left(\frac{\alpha_s}{4\pi}\right)^2\frac{C_F\alpha_s m_b^2}{4\pi}\left(\sum_{m=0}^{\infty} F^2(m)\left((m+1)\left(C_0^2 - \ln\left(\frac{2k}{\mu}\right)C_1^2 + \right.\right.\right.$$
$$\left.\left.\ln^2\left(\frac{2k}{\mu}\right)C_2^2\right) + (m+1)\Psi_1(m+2)\left(C_1^2 - 2\ln\left(\frac{2k}{\mu}\right)C_2^2\right) + I(m)C_2^2\right) +$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{m-1} F(m)F(n)\left(-\frac{n+1}{m-n}\left(C_1^2 - 2\ln\left(\frac{2k}{\mu}\right)C_2^2\right) + J(m,n)C_2^2\right) + \tag{21}$$
$$2\sum_{m=0}^{\infty} F(m)\left(\left(C_0^2 - \ln\left(\frac{2k}{\mu}\right)C_1^2 + \ln^2\left(\frac{2k}{\mu}\right)C_2^2\right) - \right.$$
$$\left.(2\gamma_E + \Psi_1(m+1))\left(C_1^2 - 2\ln\left(\frac{2k}{\mu}\right)C_2^2\right) + K(m)C_2^2\right) +$$
$$\left. 2\left(\gamma_E\ln\left(\frac{2k}{\mu}\right) + \frac{1}{2}\ln^2\left(\frac{2k}{\mu}\right)\right)\left(C_1^2 - 2\ln\left(\frac{2k}{\mu}\right)C_2^2\right) + L(k)C_2^2\right)$$

where

$$I(m) = (m+1)\left(\Psi_1^2(m+2) - \Psi_2(m+2) + 2\zeta(2) - \frac{2}{(m+1)^2}\right) -$$
$$-2(\Psi_1(m+1) + \gamma_E),$$
$$J(m,n) = 2\frac{n+1}{m-n}\left(\Psi_1(m-n) - \frac{1}{n+1} + 2\gamma_E\right) +$$
$$+2\frac{m+1}{m-n}(\Psi_1(m-n+1) - \Psi_1(m+1)),$$

$$K(m) = 2(\Psi_1(m+1) + \gamma_E)^2 + \Psi_2(m+1) - \Psi_1^2(m+1) + 2\gamma_E^2,$$
$$L(k) = -\left(\gamma_E + \frac{\pi^2}{6}\right)\ln\left(\frac{2k}{\mu}\right) - \gamma_E \ln^2\left(\frac{2k}{\mu}\right) - \frac{1}{3}\ln^3\left(\frac{2k}{\mu}\right),$$
$$\gamma_E = 0.57\ldots, \qquad \Psi_1(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad \Psi_2(x) = \Psi_1'(x)$$

and $\Gamma(x)$ is the Euler Γ-function.