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Corrections of $\mathcal{O}(\alpha\alpha_s)$ to the Decay of the Z Boson into Bottom Quarks

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Abstract

For the vertex corrections to the partial decay rate $\Gamma(Z \to b\bar{b})$ involving the top quark only the leading terms of order $\alpha\alpha_s$ in the $1/M_t$ expansion are known. In this work we compute the missing next-to-leading corrections. They turn out to be equally important as the leading term and therefore have to be included into an $\mathcal{O}(\alpha\alpha_s)$ analysis to arrive at reliable predictions.
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1 Introduction and notation

At the Large Electron Positron collider (LEP) at CERN approximately four million decays of the Z boson per experiment have been observed. Because of this enormous statistic a lot of observables have been measured with very high precision — sometimes in the region of a few per mille or even below. Also the hadronic decay of the Z boson has been measured to an accuracy of roughly 0.1% and amounts to $\Gamma_{\rm had} = 1742.4 \pm 3.1$ MeV [1].

From the theoretical side also a lot of effort has been undertaken in order to give a precise prediction for $\Gamma_{\rm had}$. The QCD corrections are known up to $\mathcal{O}(\alpha_s^3)$ both in the massless limit [2] and at $\mathcal{O}(M_q^2/s)$ [3, 4]. A complete $\mathcal{O}(\alpha_s^2)$ result [5] and the leading terms at $\mathcal{O}(\alpha_s^3)$ [6] are available for the singlet contribution (for reviews see also [7, 8]). Concerning purely electroweak corrections a complete calculation is available at one-loop level [9], whereas at two loops the leading terms in M_t are known [10]. The mixed corrections of $\mathcal{O}(\alpha\alpha_s)$ for the decay of the Z boson into the quark flavours u, d, s and c were considered in [11, 12, 13]. From these results also those $\mathcal{O}(\alpha\alpha_s)$ contributions to $\Gamma(Z \to b\bar{b})$ can be extracted where a photon or a Z boson is exchanged between the bottom quarks. For the other class of diagrams contributing to the decay into bottom quarks, namely those involving W bosons and top quarks in the loop, only the leading M_t^2 corrections [14, 15] and the $\ln M_t^2$ terms [16, 17] are at hand. In this work we close the gap and provide the constant term at next-to-leading order. In addition three more terms in the high-M_t expansion are computed. It will be demonstrated that these five terms should provide a reasonably good approximation to the full result.

It is useful to distinguish in the decay of the Z boson between universal and non-universal corrections. Universal terms are independent of the produced fermions and arise from corrections to the gauge boson propagators. The non-universal corrections, sometimes just referred to as vertex corrections, depend on the fermion species considered. Of special interest is thereby the decay into bottom quarks since in this case an additional dependence on the top quark mass appears.

The partial decay rate of the Z boson into bottom quarks can be written in the form

$$\Gamma(Z \to b\bar{b}) \;=\; \Gamma^0 \left(v_b^2 + a_b^2\right) + \delta\Gamma_{\rm univ} + \delta\Gamma_b \,, \qquad (1)$$

with $\Gamma^0 = N_c M_Z \alpha / 12 s_\Theta^2 c_\Theta^2$, $v_b = -1/2 + 2s_\Theta^2/3$ and $a_b = -1/2$. s_Θ is the sine of the weak mixing angle and $c_\Theta^2 = 1 - s_\Theta^2$. $\delta\Gamma_{\rm univ}$ represents the universal corrections and contains contributions from the transversal part of the renormalized Z boson polarization function evaluated for $q^2 = M_Z^2$, Δr, the radiative corrections entering the relation between G_F, α, M_Z and M_W, and the universal corrections arising from the $\gamma - Z$ interference. $\delta\Gamma_b$ comprises all terms directly connected to the $Zb\bar{b}$ vertex. This includes both pure QED, QCD and electroweak corrections and the mixed contributions of order $\alpha\alpha_s$. In this work we will concentrate on non-universal corrections to $\delta\Gamma_b$ where the top quark is involved, in the following denoted by[1] $\delta\Gamma_b^W$. The corresponding diagrams are pictured in

[1] Although the index W has been chosen, in a covariant gauge also the diagrams involving the charged Goldstone bosons $\phi^\pm$ have to be considered.

Fig. 1. To get the $\mathcal{O}(\alpha\alpha_s)$ corrections an internal gluon has to be attached to them in all possible ways. Analogously, we define $\delta\Gamma_b^Z$ as the contribution arising from the diagrams with internal Z boson exchange. The mass of the b quark will be neglected throughout this paper.

Whereas $\delta\Gamma_b^Z$ is finite after the Z boson contribution of the wave function renormalization of the quarks is taken into account, for $\delta\Gamma_b^W$ an additional counterterm induced by the Born and $\mathcal{O}(\alpha_s)$ result has to be added. In Feynman gauge it reads[2] [12]:

$$
\begin{aligned}
\delta\Gamma_b^W &= \delta\Gamma_b^{0,W} + \delta\Gamma_b^{\mathrm{ct},W}\,,\\
\delta\Gamma_b^{ct,W} &= \Gamma^0 \frac{1}{s_\Theta^2}\frac{\alpha}{\pi} c_\Theta^2 \Bigg\{ \frac{1}{\varepsilon}\left[-\frac{1}{6} - \frac{1}{3}\,c_\Theta^2 + \frac{\alpha_s}{\pi}\,C_F \left(-\frac{1}{8} - \frac{1}{4}\,c_\Theta^2 \right) \right] \\
&\quad - \frac{5}{18} - \frac{5}{9}\,c_\Theta^2 + \left(-\frac{1}{3} - \frac{2}{3}\,c_\Theta^2 \right) l_{\mu W} + \left(-\frac{1}{6} - \frac{1}{3}\,c_\Theta^2 \right) l_\Theta \\
&\quad + \frac{\alpha_s}{\pi}\,C_F \Bigg[-\frac{55}{48} - \frac{55}{24}\,c_\Theta^2 + \left(1 + 2\,c_\Theta^2 \right) \zeta_3 \\
&\qquad + \left(-\frac{3}{8} - \frac{3}{4}\,c_\Theta^2 \right) l_{\mu W} + \left(-\frac{1}{4} - \frac{1}{2}\,c_\Theta^2 \right) l_\Theta \Bigg] \Bigg\}\,, \qquad (2)
\end{aligned}
$$

with $\zeta_3 \approx 1.202056903$, $l_{\mu W} = \ln(\mu^2/M_W^2)$ and $l_\Theta = \ln c_\Theta^2$. $\delta\Gamma_b^{0,W}$ is the contribution from the Feynman diagrams depicted in Fig. 1.

In principle there are several options to write down an equation for $\Gamma(Z \to b\bar{b})$ containing the radiative corrections. Eq. (1) is only one possibility. Often parts of the vertex corrections are comprised into so-called effective couplings and the QCD corrections are described by a factor $(1+\alpha_s/\pi)$. This will be discussed in Section 3 where the results are presented. In the next Section some details on the calculation are given.

2 The calculation

In the following the method used for the calculation is briefly described. Instead of directly computing the two-loop vertex diagrams the three-loop polarization function of the Z boson is considered. The imaginary part then immediately leads to contributions to $\delta\Gamma_b^W$. This avoids the separate treatment of infra-red and collinear divergences and has the advantage that the advanced computational tools available for two-point functions can be used. In addition, as the top quark is much heavier than all other mass scales involved in the problem, it is tempting to perform an expansion in the inverse top quark mass. The method which provides a systematic expansion in $1/M_t$ is given by the so-called hard mass procedure (HMP) [18].

The application of the HMP requires in a first step the identification of all hard subgraphs which contain all lines carrying the large mass and whose connectivity components are one-particle-irreducible with respect to lines corresponding to light or massless particles. These subgraphs have to be expanded with respect to the small masses and the

[2] We are using dimensional regularization with space-time dimension $D = 4-2\varepsilon$, suppressing, however, the $\ln 4\pi$ and γ_E terms in all formulas.

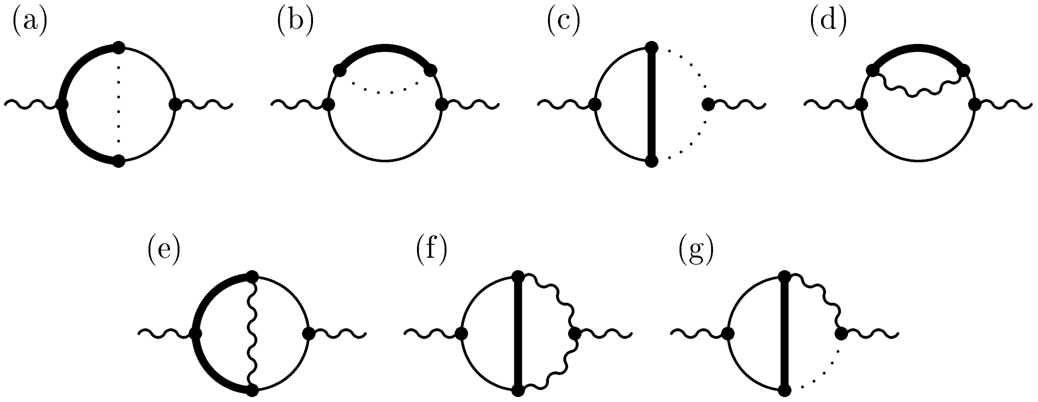

Figure 1: Diagrams contributing to $\delta\Gamma_b^W$ in $\mathcal{O}(\alpha)$. Thin lines correspond to bottom quarks, thick lines to top quarks, dotted lines to Goldstone bosons and inner wavy lines represent W bosons.

external momenta. In the full diagram all lines of the hard subgraph have to be shrunk to a point to give the so-called co-subgraph. The result of the expansion is inserted as an effective vertex. Finally the loop integrations are performed.

At two-loop level seven diagrams have to be considered. Their calculation would still be feasible by hand. At three loops, however, 69 diagrams contribute and a computation by hand is very painful especially if higher order terms in the $1/M_t$ expansion are considered. The HMP applied to the 69 initial diagrams contributing to $\delta\Gamma_b^W$ results in 234 subdiagrams which have to be expanded in their small quantities. For this reason the program package EXP written in Fortran 90 was developed. It generates all possible subgraphs of a given diagram together with some auxiliary files which allow the computation to be done automatically. The output of EXP can directly be used as input for MATAD [19] and MINCER [20] where the loop integration is performed.

It is possible to separate the diagrams at two-loop level (see Fig. 1) into two classes: If the HMP is applied to the diagrams of type (a), (b), (d) or (e) the resulting integrals factorize into massive diagrams with only one mass scale times purely massless ones with external momentum q. For the types (c), (f) and (g) the HMP also leads to a factorization of the original integral. Here, however, massive one-loop integrals with two different masses and non-vanishing external momentum have to be computed. At three-loop level, where the same classification is valid, in principle even the $\mathcal{O}(\varepsilon)$ part would be necessary. For this reason we apply the HMP again to these co-subgraphs with the conditions $M_W^2 \gg \xi_W M_W^2 \gg q^2$, or alternatively, $\xi_W M_W^2 \gg M_W^2 \gg q^2$. Here, ξ_W is the gauge parameter appearing in the W and ϕ propagators. Of course, both descriptions must lead to identical final results as the ξ_W dependence drops out at the very end. In intermediate steps, however, the expressions which have to be evaluated are different.

At the end $q^2 = M_Z^2$ has to be chosen. Then the above inequalities are seemingly inadequate. However, a closer look to the corresponding diagrams shows that actually $(2M_W)^2$, respectively $(M_W + M_t)^2$, has to be compared with q^2. The procedure is furthermore justified a posteriori as rapidly converging series are obtained from the expansions.

3 Results

Since we are interested in the virtual effect of the top quark which renders the decay of the Z boson into bottom quarks different from the one into other down-type quarks, in the following the quantity $\delta\Gamma_b^{0,W} - \delta\Gamma_d^{0,W}$ is considered. In this difference the counterterm contribution exactly cancels. This means that the renormalized partial decay rate can be written in the form

$$\delta\Gamma_b^W \;=\; \delta\Gamma_d^W + \left(\delta\Gamma_b^{0,W} - \delta\Gamma_d^{0,W}\right), \tag{3}$$

where $\delta\Gamma_d^W$ is the contribution from the diagrams involving a W boson to the partial decay rate $\Gamma(Z\to d\bar{d})$ [12]. Note that besides the $1/\varepsilon$ poles also the dependence on ξ_W drops out in the difference $\delta\Gamma_b^{0,W} - \delta\Gamma_d^{0,W}$. For convenience the following notation is introduced:

$$\delta\Gamma_{b-d}^W \;=\; \delta\Gamma_b^{0,W} - \delta\Gamma_d^{0,W} \;=\; \delta\Gamma_{b-d}^{(1)} + \frac{\alpha_s}{\pi} C_F\,\delta\Gamma_{b-d}^{(2)}\,. \tag{4}$$

For $\delta\Gamma_{b-d}^{(1)}$ we get:

$$\begin{aligned}
\delta\Gamma_{b-d}^{(1)} &= \Gamma^0 \frac{1}{s_\Theta^2}\frac{\alpha}{\pi}\times \\
&\left\{\frac{M_t^2}{M_W^2}\left[-\frac{1}{24} - \frac{1}{48}\frac{1}{y}\right]\right. \\
&+\left[\frac{857}{3240} + \frac{407}{2592}\frac{1}{y} - \frac{41}{567}\,y - \frac{731}{56700}\,y^2 - \frac{11219}{363825}\,y^3 + L_{tW}\left(-\frac{1}{8} - \frac{1}{18}\frac{1}{y}\right.\right. \\
&\quad\left.\left. - \frac{1}{36}\,y\right) + l_\Theta\left(\frac{1}{54}\frac{1}{y} - \frac{7}{270}\,y + \frac{1}{135}\,y^2 + \frac{16}{945}\,y^3\right) + \mathcal{O}(y^4)\right] \\
&+\frac{M_W^2}{M_t^2}\left[\frac{749}{12960} + \frac{61}{432}\frac{1}{y} - \frac{10033}{22680}\,y + \frac{947}{22680}\,y^2 + \frac{298}{4455}\,y^3 + \frac{11038}{405405}\,y^4\right. \\
&\quad\left. + L_{tW}\left(-\frac{23}{72} - \frac{5}{24}\frac{1}{y} + \frac{2}{9}\,y + \frac{1}{18}\,y^2\right) + \mathcal{O}(y^5)\right] \\
&+\left(\frac{M_W^2}{M_t^2}\right)^2\left[-\frac{3707}{6480} + \frac{43}{288}\frac{1}{y} - \frac{15251}{16200}\,y + \frac{92531}{56700}\,y^2 + \frac{137}{22275}\,y^3 - \frac{18464}{135135}\,y^4\right. \\
&\quad\left. - \frac{20896}{405405}\,y^5 + L_{tW}\left(-\frac{1}{9} - \frac{17}{48}\frac{1}{y} + \frac{19}{20}\,y - \frac{5}{9}\,y^2 - \frac{2}{15}\,y^3\right) + \mathcal{O}(y^6)\right] \\
&+\left(\frac{M_W^2}{M_t^2}\right)^3\left[-\frac{22399}{12960} + \frac{65}{432}\frac{1}{y} + \frac{1063}{7560}\,y + \frac{285163}{48600}\,y^2 - \frac{2425274}{467775}\,y^3\right. \\
&\quad - \frac{96794}{868725}\,y^4 + \frac{127808}{405405}\,y^5 + \frac{753776}{6891885}\,y^6 + L_{tW}\left(\frac{215}{216} - \frac{109}{216}\frac{1}{y} + \frac{43}{27}\,y\right. \\
&\quad\left.\left. - \frac{371}{90}\,y^2 + \frac{8}{5}\,y^3 + \frac{16}{45}\,y^4\right) + \mathcal{O}(y^7)\right]
\end{aligned}$$

$$+\mathcal{O}\left(\left(\frac{M_W^2}{M_t^2}\right)^4\right)\Bigg\}, \tag{5}$$

with $y = 1/(4c_\Theta^2)$, $\zeta_2 = \pi^2/6$, $L_{tW} = \ln(M_t^2/M_W^2)$ and l_Θ as in Eq. (2). Note that the successive application of the HMP to the diagrams of class $(c), (f)$ and (g) (see Fig. 1) leads to an expansion in $1/c_\Theta^2$. However, since a W boson in the final state is always accompanied by either another W boson or a top quark, the use of the variable y defined above seems to be more adequate. We have checked our result for the one-loop terms of $\delta\Gamma_b^W$ against the expansion of the full one-loop result [9] and complete agreement was found. Numerically, the expansion approximates the full answer to 0.01% accuracy for a reasonable choice of m_t, M_W and M_Z. This puts quite some confidence into the reliability of the corresponding two-loop terms.

Nevertheless, two remarks concerning the expansion in $1/c_\Theta^2$ are in order: First, note that the above series is nested and therefore each order in $1/M_t^2$ produces an additional power in y. To demonstrate the quality of the convergence all available terms are displayed. Second, one recognizes that the coefficients of the logarithms seem to be truncated series in $1/c_\Theta^2$, which suggests that they might be exact. These remarks are also taken over to the $\mathcal{O}(\alpha\alpha_s)$ corrections given by

$$\begin{aligned}
\delta\Gamma_{b-d}^{(2)} &= \Gamma^0 \frac{1}{s_\Theta^2}\frac{\alpha}{\pi}\times \\
&\Bigg\{\frac{M_t^2}{M_W^2}\left[-\frac{1}{32}-\frac{1}{64}\frac{1}{y}+\zeta_2\left(\frac{1}{16}+\frac{1}{32}\frac{1}{y}\right)\right] \\
&+\left[\frac{75857}{466560}+\frac{7}{32}\frac{1}{y}-\frac{121895171}{204120000}y+\frac{4262581}{10206000}y^2-\frac{3177101006}{4011170625}y^3\right. \\
&\quad+\zeta_2\left(\frac{173}{1296}+\frac{67}{2592}\frac{1}{y}+\frac{53}{324}y\right)+\zeta_3\left(-\frac{1}{18}\frac{1}{y}+\frac{7}{90}y-\frac{1}{45}y^2-\frac{16}{315}y^3\right) \\
&\quad+L_{tW}\left(-\frac{757}{7776}-\frac{331}{7776}\frac{1}{y}-\frac{95}{3888}y\right)+l_\Theta^2\left(-\frac{103}{2592}-\frac{1}{81}\frac{1}{y}+\frac{1}{300}y\right. \\
&\quad\left.-\frac{103}{1080}y^2-\frac{5314}{33075}y^3\right)+l_\Theta\left(-\frac{527}{7776}+\frac{11}{288}\frac{1}{y}-\frac{1489}{30375}y+\frac{1081}{9720}y^2\right. \\
&\quad\left.\left.-\frac{3338578}{10418625}y^3\right)+\mathcal{O}(y^4)\right] \\
&+\frac{M_W^2}{M_t^2}\left[-\frac{384287}{1555200}+\frac{47}{3456}\frac{1}{y}-\frac{5741993}{5443200}y-\frac{217997}{226800}y^2+\frac{149}{1215}y^3+\frac{5519}{110565}y^4\right. \\
&\quad+L_{tW}\left(-\frac{83083}{155520}-\frac{1819}{5184}\frac{1}{y}+\frac{15017}{38880}y+\frac{3977}{38880}y^2\right)+l_\Theta\left(-\frac{3343}{155520}-\frac{7}{1296}\frac{1}{y}\right. \\
&\quad\left.\left.-\frac{823}{38880}y+\frac{17}{38880}y^2\right)+\zeta_2\left(\frac{257}{864}+\frac{175}{864}\frac{1}{y}+\frac{13}{144}y+\frac{11}{18}y^2\right)+\mathcal{O}(y^5)\right] \\
&+\left(\frac{M_W^2}{M_t^2}\right)^2\left[\frac{470213}{1166400}-\frac{6119}{31104}\frac{1}{y}+\frac{54467207}{105840000}y-\frac{640160203}{158760000}y^2-\frac{3791409547}{982327500}y^3\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{44597}{173745}\,y^4 - \frac{38284}{405405}\,y^5 + \zeta_2\left(- \frac{239}{324} + \frac{10}{27}\frac{1}{y} - \frac{14539}{10800}\,y + \frac{4769}{1080}\,y^2 \right.\\
&\left. + \frac{4838}{2025}\,y^3\right) + L_{tW}\left(- \frac{14791}{38880} - \frac{635}{864}\frac{1}{y} + \frac{2935841}{1701000}\,y - \frac{1769611}{1701000}\,y^2 - \frac{19039}{70875}\,y^3\right)\\
&+ L_{tW}\, l_\Theta \left(\frac{1}{36} + \frac{1}{144}\frac{1}{y} + \frac{1}{36}\,y\right) + l_\Theta \left(- \frac{623}{19440} - \frac{7}{864}\frac{1}{y} - \frac{105443}{3402000}\,y \right.\\
&\left.\left. + \frac{283}{212625}\,y^2 - \frac{31}{283500}\,y^3\right) + L_{tW}^2 \left(\frac{1}{36} + \frac{1}{144}\frac{1}{y} + \frac{1}{36}\,y\right) + \mathcal{O}(y^6)\right]\\
&+ \left(\frac{M_W^2}{M_t^2}\right)^3 \left[\frac{1103711}{388800} - \frac{165325}{373248}\frac{1}{y} + \frac{608169487}{190512000}\,y - \frac{91635792023}{6429780000}\,y^2 \right.\\
&- \frac{345910967113}{17681895000}\,y^3 - \frac{3664430299069}{229864635000}\,y^4 + \frac{655346}{1216215}\,y^5 + \frac{238846}{1342575}\,y^6\\
&+ \zeta_2\left(- \frac{9497}{2592} + \frac{1397}{2592}\frac{1}{y} - \frac{6163}{3240}\,y + \frac{52021}{3240}\,y^2 + \frac{163}{25}\,y^3 + \frac{6392}{675}\,y^4\right)\\
&+ L_{tW}\left(\frac{233569}{155520} - \frac{36293}{31104}\frac{1}{y} + \frac{1328861}{453600}\,y - \frac{169446869}{20412000}\,y^2 + \frac{1726367}{637875}\,y^3 \right.\\
&\left. + \frac{62576}{91125}\,y^4\right) + L_{tW}\, l_\Theta\left(\frac{5}{81} + \frac{5}{324}\frac{1}{y} + \frac{5}{81}\,y\right) + l_\Theta\left(- \frac{5419}{155520} - \frac{11}{1296}\frac{1}{y} \right.\\
&\left. - \frac{51157}{1360800}\,y - \frac{78299}{20412000}\,y^2 - \frac{148}{637875}\,y^3 + \frac{7}{182250}\,y^4\right)\\
&\left. + L_{tW}^2\left(\frac{5}{81} + \frac{5}{324}\frac{1}{y} + \frac{5}{81}\,y\right) + \mathcal{O}(y^7)\right]\\
&\left. + \mathcal{O}\left(\left(\frac{M_W^2}{M_t^2}\right)^4\right)\right\}. \qquad (6)
\end{aligned}
$$

For the definition of the top quark mass, M_t, the on-shell scheme has been adopted. With the help of the equation

$$
M_t \;=\; m_t(\mu)\left[1 + \frac{\alpha_s(\mu)}{\pi} C_F \left(1 + \frac{3}{4}\ln\frac{\mu^2}{m_t^2(\mu)}\right)\right] \qquad (7)
$$

the transformation to the $\overline{\mathrm{MS}}$ scheme can be performed.

There are several checks for the correctness of our results. First of all, our new calculation with the automated HMP provides an independent check of the M_t^2 and the $\ln M_t^2$ term. The terms up to (and including) order $1/M_t^2$ and $1/c_\Theta^2$ were calculated with arbitrary gauge parameters ξ_W and ξ_S for the electroweak sector and QCD, respectively. We could check that to each order in the $1/M_t$ expansion ξ_W drops out separately. Whereas for the M_t^2 and $1/M_t^2$ terms this happens after taking the sum of all contributing diagrams, for the $(M_t^2)^0$ order only the difference $\delta\Gamma_b^{0,W} - \delta\Gamma_d^{0,W}$ renders the result gauge invariant[3]. ξ_S already drops out if only the sum of diagrams contributing to the separate classes (see

[3]It should be noted that we partly had to repeat the calculation done in [12] for arbitrary gauge parameter ξ_W.

Fig. 1) is considered. The $1/M_t^4$ and $1/M_t^6$ corrections and the higher order terms in the $1/c_\Theta^2$ expansion were computed in Feynman gauge as then the calculation is much faster. Note, that in Eq. (6) where all parameters are expressed in the on-shell scheme the explicit μ dependence drops out which is also a welcome check for our calculation.

Let us now discuss the numerical relevance of the newly computed terms. Using $s_\Theta^2 = 0.223$ the corrections induced by W-bosons read in the on-shell scheme:

$$
\begin{aligned}
\delta\Gamma^W_{b-d} &= \Gamma^0 \frac{1}{s_\Theta^2}\frac{\alpha}{\pi}\times \\
&\Bigg\{ -0.11\,\frac{M_t^2}{M_W^2} + 0.71 - 0.31\,L_{tW} + (0.36 - 0.89\,L_{tW})\,\frac{M_W^2}{M_t^2} \\
&+ (-0.24 - 0.97\,L_{tW})\left(\frac{M_W^2}{M_t^2}\right)^2 + (-0.78 - 0.43\,L_{tW})\left(\frac{M_W^2}{M_t^2}\right)^3 \\
&+ \frac{\alpha_s}{\pi}\Bigg[0.24\,\frac{M_t^2}{M_W^2} + 1.21 - 0.32\,L_{tW} + (1.40 - 1.99\,L_{tW})\,\frac{M_W^2}{M_t^2} \\
&+ \left(0.37 - 2.99\,L_{tW} + 0.08\,L_{tW}^2\right)\left(\frac{M_W^2}{M_t^2}\right)^2 \\
&+ \left(-1.08 - 2.64\,L_{tW} + 0.17\,L_{tW}^2\right)\left(\frac{M_W^2}{M_t^2}\right)^3\Bigg]\Bigg\} + \mathcal{O}\left(\left(\frac{M_W^2}{M_t^2}\right)^4\right). \qquad (8)
\end{aligned}
$$

Using Eq. (7) one may transform the top mass, M_t, to the $\overline{\mathrm{MS}}$ scheme. For the renormalization scale μ explicitly appearing in the resulting expression, $\delta\bar{\Gamma}_b^W$, we adopt the choice $\mu^2 = M_Z^2$. One finds (m_t denotes the $\overline{\mathrm{MS}}$ top mass, and $l_{tW} = \ln(m_t^2/M_W^2)$):

$$
\begin{aligned}
\delta\bar{\Gamma}^W_{b-d} &= \Gamma^0 \frac{1}{s_\Theta^2}\frac{\alpha}{\pi}\times \\
&\Bigg\{ -0.11\,\frac{m_t^2}{M_W^2} + 0.71 - 0.31\,l_{tW} + (0.36 - 0.89\,l_{tW})\,\frac{M_W^2}{m_t^2} \\
&+ (-0.24 - 0.97\,l_{tW})\left(\frac{M_W^2}{m_t^2}\right)^2 + (-0.78 - 0.43\,l_{tW})\left(\frac{M_W^2}{m_t^2}\right)^3 \\
&+ \frac{\alpha_s}{\pi}\Bigg[(-0.09 + 0.21\,l_{tW})\,\frac{m_t^2}{M_W^2} + 0.24 + 0.30\,l_{tW} \\
&+ \left(-2.57 + 3.34\,l_{tW} - 1.78\,l_{tW}^2\right)\frac{M_W^2}{m_t^2} \\
&+ \left(-1.16 + 4.12\,l_{tW} - 3.80\,l_{tW}^2\right)\left(\frac{M_W^2}{m_t^2}\right)^2 \\
&+ \left(4.99 - 2.37\,l_{tW} - 2.41\,l_{tW}^2\right)\left(\frac{M_W^2}{m_t^2}\right)^3\Bigg]\Bigg\} + \mathcal{O}\left(\left(\frac{M_W^2}{m_t^2}\right)^4\right). \qquad (9)
\end{aligned}
$$

One observes that for realistic values of M_Z and M_t, respectively m_t, the constant at next-to-leading order dominates over the $\ln M_t^2$ term known before. Furthermore, the size of the contributions from the individual $1/M_t^2$ orders is roughly the same in both schemes. It is remarkable that at one-loop level the corrections arising from the $1/M_t^2$ terms are of similar size than the one from next-to-leading order, however, the signs are different. The higher order corrections in $1/M_t$ are smaller, which means that effectively only the leading M_t^2 term remains. Proceeding to two loops the situation is similar: Starting at $\mathcal{O}(1/M_t^2)$ the sign is opposite as compared to the leading terms and a large cancellation takes place. Here, the $1/M_t^4$ term is still comparable with the $1/M_t^2$ contribution. The $1/M_t^6$ term, however, is smaller by more than a factor 1/4, which suggests that the presented terms should provide a reasonable approximation to the full result.

For $M_t = 175$ GeV, $M_Z = 91.19$ GeV, $\alpha = 1/129$ and $\alpha_s(M_Z) = 0.120$ we get:

$$\delta\Gamma_{b-d}^W = (-5.69 - 0.79 + 0.50 + 0.06) \text{ MeV} = -5.92 \text{ MeV}. \tag{10}$$

The first two numbers in Eq. (10) correspond to the $\mathcal{O}(\alpha)$, the second two to the $\mathcal{O}(\alpha\alpha_s)$ corrections. Each of these contributions is again separated into the M_t^2 terms and the sum of the subleading ones. For the diagrams containing a Z boson [12] the numerical value reads:

$$\delta\Gamma_b^Z = (0.52 - 0.01) \text{ MeV} = 0.51 \text{ MeV}. \tag{11}$$

One can see that even the contributions of the subleading terms of diagrams arising from a W boson exchange are more important than the corrections resulting from $\delta\Gamma_b^Z$.

Let us finally also give the net effect of the non-factorizable corrections. Adopting a notation analogue to Eq. (4) and using the expansion for $\delta\Gamma_d^W$ in the limit of small y [12][4] the difference to the result where "naive" factorization is assumed reads

$$\frac{\alpha_s}{\pi}\left(C_F\,\delta\Gamma_b^{(2),W} - \delta\Gamma_b^{(1),W}\right) = 0.68 \text{ MeV}. \tag{12}$$

This is comparable both to the error on the present experimental value of $\Gamma_{\rm had}$ and even to the one-loop corrections from the Z boson diagrams (see Eq. (11)). For $\Gamma(Z \to b\bar{b})$ the factorization is often performed such that the leading M_t^2 term is reproduced correctly. If this is taken into account the deviation corresponding to Eq. (12) reduces to -0.04 MeV which is still larger than the mixed $\mathcal{O}(\alpha\alpha_s)$ corrections induced by internal Z bosons (second number in Eq. (11)).

To conclude, the missing non-universal piece to the decay of the Z boson into bottom quarks to $\mathcal{O}(\alpha\alpha_s)$ has been computed as an expansion for large top quark mass. It turns out that the first few terms in the series are of comparable magnitude. Their inclusion in an experimental analysis is thus mandatory. The higher terms, however, decrease rapidly which justifies the expectation that the truncated series approximates the full answer fairly well, as is also indicated by the behaviour of the one-loop result. The results are presented in a form which allows a simple implementation into program libraries for the description of the Z line shape (see [7] and references therein).

[4] We had to increase the depth of the expansion in y for these terms.

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References

[1] G. Altarelli, Report Nos. CERN-TH-97-278 and hep-ph/9710434;
G. Quast, Proceedings of the *International Europhysics Conference on High Energy Physics*, Israel, (to appear).

[2] S.G. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B 259** (1991) 144;
L.R. Surguladze and M.A. Samuel, *Phys. Rev. Lett.* **66** (1991) 560; (E) ibid., 2416;
K.G. Chetyrkin, *Phys. Lett.* **B 391** (1997) 402.

[3] K.G. Chetyrkin and J.H. Kühn, *Phys. Lett.* **B 248** (1990) 359.

[4] K.G. Chetyrkin and J.H. Kühn, *Phys. Lett.* **B 406** (1997) 102.

[5] B.A. Kniehl and J.H. Kühn, *Phys. Lett.* **B 224** (1989) 229; *Nucl. Phys.* **B 329** (1990) 547.

[6] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, *Phys. Lett.* **B 320** (1994) 159; *Nucl. Phys.* **B 438** (1995) 278;
K.G. Chetyrkin and O.V. Tarasov, *Phys. Lett.* **B 327** (1994) 114.

[7] D. Bardin et al., *Reports of the working group on precision calculations for the Z resonance*, D. Bardin, W. Hollik, G. Passarino, (eds.), CERN 95-03, 31 Mar 1995.

[8] K.G. Chetyrkin, J.H. Kühn and A. Kwiatkowski, *Phys. Rep.* **277** (1996) 189.

[9] A. Akhundov, D. Bardin and T. Riemann, *Nucl. Phys.* **B 276** (1986) 1;
W. Beenakker and W. Hollik, *Z. Phys.* **C 40** (1988) 141;
J. Bernabeu, A. Pich and A. Santamaria, *Phys. Lett.* **B 200** (1988) 569;
B.W. Lynn and R.G. Stuart *Phys. Lett.* **B 252** (1990) 676.

[10] R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci and A. Vicere, *Phys. Lett.* **B 288** (1992) 95; (E) ibid. **B 312** 511; *Nucl. Phys.* **B 409** (1993) 105;
J. Fleischer, O.V. Tarasov and F. Jegerlehner, *Phys. Lett.* **B 319** (1993) 249.

[11] A.L. Kataev, *Phys. Lett.* **B 287** (1992) 209.

[12] A. Czarnecki and J.H. Kühn, *Phys. Rev. Lett.* **77** (1996) 3955; hep-ph/9608366 (revised version).

[13] A. Czarnecki and K. Melnikov, *Phys. Rev.* **D 56** (1997) 1638.

[14] J. Fleischer, F. Jegerlehner, P. Rączka and O.V. Tarasov, *Phys. Lett.* **B 293** (1992) 437;
G. Buchalla and A. Buras, *Nucl. Phys.* **B 398** (1993) 285;
G. Degrassi, *Nucl. Phys.* **B 407** (1993) 271.

[15] K.G. Chetyrkin, A. Kwiatkowski and M. Steinhauser, *Mod. Phys. Lett.* **A 8** (1993) 2785.

[16] A. Kwiatkowski and M. Steinhauser, *Phys. Lett.* **B 344** (1995) 359.

[17] S. Peris and A. Santamaria, *Nucl. Phys.* **B 445** (1995) 252.

[18] For a review see e.g.: V.A. Smirnov, *Mod. Phys. Lett.* **A 10** (1995) 1485.

[19] M. Steinhauser, Ph.D. thesis, Karlsruhe University (Shaker Verlag, Aachen, 1996).

[20] S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, Preprint NIKHEF-H/91-18 (1991).