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# Beta functions and anomalous dimensions up to three loops

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**Abstract**

We derive an algorithm for automatic calculation of perturbative $\beta$-functions and anomalous dimensions in any local quantum field theory with canonical kinetic terms. The infrared rearrangement is performed by introducing a common mass parameter in all the propagator denominators. We provide a set of explicit formulae for all the necessary scalar integrals up to three loops.

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## 1. Introduction

Renormalization group equations are a fundamental tool in modern quantum field theory. In phenomenological applications, their evaluation with sufficient accuracy often requires finding multiloop contributions to $\beta$-functions and anomalous dimensions. In the present paper, we describe a simple algorithm for calculating these quantities in the framework of dimensional regularization and the $MS$ (or $\overline{MS}$) scheme.

In a mass-independent renormalization scheme, $\beta$-functions and anomalous dimensions are simply related to coefficients at counterterms which renormalize ultraviolet divergences. A remarkable feature of the $MS$-scheme is the fact that in its framework all the UV counterterms are polynomial *both* in momenta *and* in masses [1].[2] Consequently, a certain expansion in external momenta and masses can be performed before integration over loop momenta, which radically simplifies the integrals one needs to calculate.

The main difficulty in this procedure is appearance of spurious infrared divergences. The classical method of avoiding them is called "infrared rearrangement" [2, 3]. It amounts to adding artificial masses or external momenta in certain lines of a given Feynman diagram before the expansion in masses and true external momenta is made. The artificial external momenta have to be introduced in such a way that all spurious infrared divergences are removed, and the resulting Feynman integrals are calculable. Satisfying these two requirements is rather cumbersome in practical multiloop calculations. In addition, the condition that the IR divergences do not appear restricts considerably the power of the approach, since for complicated diagrams this requirement prevents one from reducing a given Feynman diagram to a simpler one.

The latter problem was completely solved with elaborating a special technique of subtraction of IR divergences — the $R^*$-operation [4]. This method allows one to express (though in a rather involved way) the UV counterterm of every (h+1)-loop Feynman integral in terms of divergent and finite parts of some properly constructed h-loop massless propagators. Unfortunately, in practical applications, the use of the $R^*$-operation requires either many manipulations with individual diagrams or resolving a lot of non-trivial problem-dependent combinatorics (see, e.g. [5, 6]).

In our approach, the infrared rearrangement is performed by introducing an artificial mass rather than an artificial external momentum. A single mass parameter is added to

[2] In any meaningful renormalization prescription, counterterms are polynomial in external momenta, but not necessarily in masses.

each denominator of a propagator in each Feynman diagram. Consequently, no spurious IR divergences can appear. Next, an expansion in all the particle masses (except, of course, the auxiliary one) and external momenta is performed. The integrals one is left with have relatively simple form: They are completely massive tadpoles, i.e. Feynman integrals without external momenta and with only a single mass inserted in all the propagators. As a result, the problem of evaluating h-loop UV counterterms eventually reduces to a computation of divergent parts of h-loop completely massive tadpoles.

At two loops, simple formulae for such Feynman integrals have been known since long ago [7]. However, no explicit formulae for three-loop massive tadpoles have been published so far. The available recursion algorithms [8, 9] based on the integration by parts method [10, 11] are quite involved.

The basic idea of our algorithm is to determine the pole part of a massive tadpole by expanding a properly chosen two-loop sub-integral with respect to its large external momentum being a loop momentum in the initial three-loop integral. Eventually, we have been able to construct relatively simple *explicit* formulae for all the necessary three-loop scalar integrals.

The algorithm described in the present paper was used at the two loop [12] and three loop [13] levels for calculating QCD anomalous dimensions of effective operators mediating $B \to X_s \gamma$ decay.

In principle, our method is applicable at the four-loop level, too. In this case, the problem eventually amounts to expanding a three-loop massive sub-integral of the propagator type with respect to its large external momentum. The algorithm for calculation of such three-loop integrals has been known since long ago (for a review see [14]) and its computer algebra implementation has been recently achieved [15].

Very recently, an alternative algorithm was developed by van Ritbergen, Vermaseren and Larin, and applied for evaluating four-loop contributions to the QCD $\beta$ function and quark mass anomalous dimension [16]. Their approach amounts to using an identical to ours version of the IR rearrangement which reduces the calculation of UV renormalization constants to calculation of massive tadpoles. The difference appears at the stage of tadpole evaluation. The authors of [16] have succeeded in creating "special routines (...) to efficiently evaluate 4-loop massive bubble integrals up to pole parts in $\epsilon$ and correspondingly of the 3-loop massive bubbles to finite parts." Eventually, all the diagrams have been reduced to two master ones.

Our paper is organized as follows: In the next section, we give general arguments which justify the use of an artificial mass parameter as an infrared regulator in all the propagators, including propagators of massless gauge bosons. This is allowed so long as we are interested only in the UV-divergent parts of regularized Green's functions (with all UV *sub*divergences being pre-subtracted). In section 3, we describe our algorithm for evaluating scalar integrals up to three loops. In section 4, we present some more details concerning calculation of nontrivial three-loop integrals. Section 5 contains two examples of relations between renormalization constants and $\beta$ functions or anomalous dimensions up to three loops. Appendix A is devoted to reduction of tensor integrals to scalar ones. Appendix B summarizes expressions for "trivial" integrals, i.e. the ones which reduce to products of lower-loop integrals. Appendix C describes the expansion of one-loop self-energy integrals at large external momentum, which constitutes an essential element in calculating nontrivial three-loop integrals. Finally, appendix D contains a useful relation between tensor and scalar one-loop integrals in different numbers of dimensions.

## 2. Decomposition of propagators

The starting point of our procedure is a certain exact decomposition of propagators. For a scalar propagator belonging to a given Feynman diagram, it has the following form:

$$\frac{1}{(q+p)^2 - M^2} = \frac{1}{q^2 - m^2} + \frac{M^2 - p^2 - 2qp - m^2}{q^2 - m^2} \frac{1}{(q+p)^2 - M^2}. \tag{1}$$

Here, $p$ is a linear combination of external momenta in the considered diagram, $q$ stands for a linear combination of loop momenta, and $M$ denotes the mass of the particle. The artificial mass parameter $m$ is introduced to regularize spurious infrared divergences. It is the same in all the propagators and all the diagrams.

The contribution of the considered propagator to the overall degree of divergence of a diagram is $\Delta\omega = -2$. The decomposition has been performed in such a way that the first simple term in the r.h.s. of eqn. (1) gives $\Delta\omega = -2$, while the second, more complicated term gives $\Delta\omega = -3$. Moreover, the very last term in eqn. (1) has the same form as the original propagator. Thus, we can decompose it in an identical way. Doing so several times, we decompose the original propagator into a sum of terms with very simple denominators (depending only on loop momenta and the mass parameter $m$), and a more complicated term whose contribution to the overall degree of divergence is arbitrarily low negative. For instance, after three steps of decomposition, the exact expression for the original propagator reads

$$
\begin{aligned}
\frac{1}{(q+p)^2-M^2} \quad = \quad & \frac{1}{q^2-m^2}+\frac{M^2-p^2-2qp}{(q^2-m^2)^2}+\frac{(M^2-p^2-2qp)^2}{(q^2-m^2)^3} \\
& -\frac{m^2}{(q^2-m^2)^2}+\frac{m^4-2m^2(M^2-p^2-2qp)}{(q^2-m^2)^3} \\
& +\frac{(M^2-p^2-2qp-m^2)^3}{(q^2-m^2)^3[(q+p)^2-M^2]}.
\end{aligned} \tag{2}
$$

Here, the last term gives $\Delta\omega = -5$ contribution to the overall degree of divergence of a diagram.

In the following, we shall assume that the theory we consider is given by an (effective) lagrangian which does not contain non-negligible operators of arbitrarily high dimension, i.e. we assume that dimensionality of our operators is bounded from above. In such a case, any particular Green's function has a certain maximal degree of divergence. Consequently, we can always perform so many steps in the propagator decomposition, that the overall degree of divergence of any diagram in this Green's function would become negative if any of its propagators was replaced by the last term in the decomposition. We are then allowed to drop the last term in each propagator decomposition. It does not affect the UV-divergent part of the Green's function (*after* subtraction of subdivergences).

It is important to note that each term in the propagator decomposition satisfies the criteria a full propagator should satisfy in the proof of Weinberg's theorem [17]. This allows to apply degree-of-divergence arguments for diagrams where propagators are replaced by particular terms in their decomposition.

A further simplification can be achieved by noticing that terms containing $m^2$ in the numerators (like the second line in the r.h.s. of eqn. (2)) contribute only to such UV-divergent terms which are proportional to $m^2$. These terms are local after subtraction of subdivergences. They must precisely cancel similar terms originating from integrals with no $m^2$ in propagator numerators. No dependence on $m^2$ can remain after performing the whole calculation, because the propagators are decomposed exactly, i.e. they are actually independent of $m^2$. This observation allows to avoid calculating integrals with $m^2$ in propagator numerators. Instead of calculating them, one can just replace them by local counterterms proportional to $m^2$ which cancel the corresponding (sub)divergences in integrals with no $m^2$ in propagator numerators. In effect, the practical calculation is made only with propagators replaced by such terms in the decomposition which contain no $m^2$ in the numerators (like the first line in the r.h.s. of eqn. (2)). Nevertheless, the final results for the divergent parts of Green's functions are precisely the same as if the full propagators were used (*after* subtraction of subdivergences).

In our earlier paper [12], we gave somewhat different arguments for using a single mass parameter as an infrared regulator in calculating $\beta$ functions and anomalous dimensions. The present considerations might be more convincing, because the propagator decomposition we discuss here is exact. Thus, $m^2$ can be kept arbitrary all the time. One does not need to consider the $m^2 \to 0$ limit and worry about its commutativity with Feynman integration.

For particles with spin other than zero, the decomposition is applied only to denominators of their propagators, provided they are the same as in the scalar propagator. Our algorithm is not applicable in theories where kinetic terms differ from the canonical ones, as e.g. in the Heavy Quark Effective Theory [18].

As we have explained, one does not need to calculate Feynman integrals containing $m^2$ in propagator numerators, so long as extra counterterms proportional to $m^2$ are introduced. Such counterterms may not preserve symmetries of the theory. Fortunately, the number of these counterterms is usually rather small, because their dimension must be at least twice smaller than the maximal dimension of operators in the considered (effective) lagrangian. For instance, the QCD lagrangian is built out of operators of dimension less or equal 4. There is only a single possible gauge-noninvariant counterterm of dimension 2. It reads

$$\frac{1}{2} Z_x m^2 G^a_\mu G^{a\,\mu}, \tag{3}$$

i.e. it looks like a "gluon mass" counterterm.[3] At one loop, we find (using the Feynman–'t Hooft gauge and the $MS$ scheme in $D = 4 - 2\epsilon$ dimensions)

$$Z_x = -\frac{g^2}{16\pi^2\epsilon}(N + 2f), \tag{4}$$

where $N$ is the number of colors and $f$ is the number of active flavors. This counterterm cancels gauge-noninvariant pieces of integrals with no $m^2$ in propagator numerators.

After dropping the last term in the propagator decomposition, the Feynman integrands one is left with depend only polynomially on particle masses and external momenta. These quantities can be factorized out. It remains to calculate integrals depending only on loop

[3]The "ghost mass" counterterm does not arise in the Feynman–'t Hooft gauge, due to the structure of the ghost-gluon vertex.

momenta and the artificial mass parameter $m^2$. At one loop, the generic integral reads

$$\int \frac{d^D q}{(2\pi)^D} \frac{q_{\mu_1} \ldots q_{\mu_k}}{(q^2 - m^2)^n}. \tag{5}$$

Integrals arising at more loops are slightly more complicated, because they involve several loop momenta. Nevertheless, reducing any such integral to scalar integrals can be easily performed by contracting it with various products of metric tensors and solving the resulting system of linear equations. We have written a Mathematica [19] code which performs such a reduction up to three loops, for an arbitrary number of free Lorentz indices. Some elements of this procedure are outlined in appendix A.

After the reduction of tensor integrals is performed, one is left with relatively small number of scalar integrals to calculate. It is convenient to use the euclidean metric in discussing their evaluation. The euclidean integrals arising at one, two and three loops are respectively as follows

$$I^{(1)}_n = m^{-D+2n}\, \pi^{-\frac{D}{2}} \int d^D q \frac{1}{[q^2 + m^2]^n}, \tag{6}$$

$$I^{(2)}_{n_1 n_2 n_3} = m^{-2D+2\Sigma n_i}\, \pi^{-D} \int d^D q_1\, d^D q_2 \frac{1}{[q_1^2 + m^2]^{n_1} [q_2^2 + m^2]^{n_2} [(q_1 - q_2)^2 + m^2]^{n_3}}, \tag{7}$$

$$I^{(3)}_{n_1 n_2 n_3 n_4 n_5 n_6} = m^{-3D+2\Sigma n_i}\, \pi^{-\frac{3D}{2}} \int d^D q_1\, d^D q_2\, d^D q_3 \times$$

$$\times \frac{1}{[q_1^2 + m^2]^{n_1} [q_2^2 + m^2]^{n_2} [q_3^2 + m^2]^{n_3} [(q_2 - q_3)^2 + m^2]^{n_4} [(q_3 - q_1)^2 + m^2]^{n_5} [(q_1 - q_2)^2 + m^2]^{n_6}}. \tag{8}$$

The chosen normalization makes them dimensionless. The integrals can be represented by scalar vacuum diagrams displayed in fig. 1 with propagators raised to arbitrary integer powers $n_i$. The algorithm for their evaluation is described in the next section.

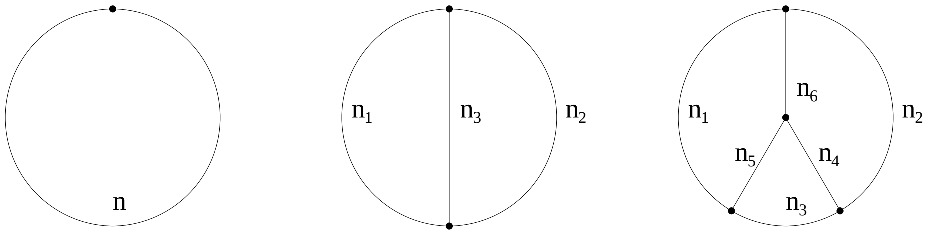


Figure 1: Graphical representation of the integrals given in eqns. (6)–(8)

## 3. The algorithm for evaluation of scalar integrals.

In this section, we assume we are interested in evaluating three-loop $\beta$-functions or anomalous dimensions. We use the MS scheme with $D = 4 - 2\epsilon$ dimensions. We need to be able to evaluate $I^{(3)}_{n_1n_2n_3n_4n_5n_6}$ up to $\mathcal{O}(\frac{1}{\epsilon})$, $I^{(2)}_{n_1n_2n_3}$ up to $\mathcal{O}(1)$ and $I^{(1)}_{n}$ up to $\mathcal{O}(\epsilon)$. The latter integral is known exactly from textbooks [20]

$$I^{(1)}_n = \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)}. \tag{9}$$

The two-loop integral $I^{(2)}_{n_1n_2n_3}$ is totally symmetric under permutations of its indices. It reduces to a product of one-loop integrals when at least one of the indices is nonpositive (see appendix B for explicit formulae). On the other hand, when all the indices are positive, it can be found from the following relations [21]:

$$\begin{aligned} I^{(2)}_{(n_1+1)n_2n_3} = \frac{1}{3n_1}\{(3n_1-D)I^{(2)}_{n_1n_2n_3} &+ n_2[I^{(2)}_{(n_1-1)(n_2+1)n_3} - I^{(2)}_{n_1(n_2+1)(n_3-1)}] \\ &+ n_3[I^{(2)}_{(n_1-1)n_2(n_3+1)} - I^{(2)}_{n_1(n_2-1)(n_3+1)}]\} \end{aligned} \tag{10}$$

and

$$I^{(2)}_{111} = \frac{\Gamma(1+\epsilon)^2}{(1-\epsilon)(1-2\epsilon)}[-\frac{3}{2\epsilon^2}+\frac{27}{2}S_2] + \mathcal{O}(\epsilon), \tag{11}$$

where

$$S_2 = -\frac{4}{9\sqrt{3}}\int_0^{\pi/3} dx \ln(2\sin\frac{x}{2}) \simeq 0.2604341. \tag{12}$$

The recursion relation (10) holds for $n_i \geq 1$. It can be derived with use of integration by parts. The sum of indices in the integral on its l.h.s. is bigger than the sum of indices in each of the integrals on its r.h.s.. Thus, the recursion can be programmed into a computer algebra code just as it stands. Two-loop integrals one usually encounters in practice are then found within a fraction of a second.

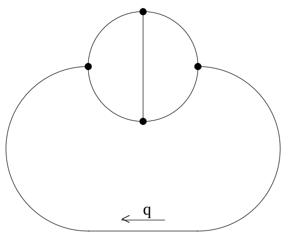


Figure 2: Graphical representation of the integral $I^{(3)}_{111111}$. It is equivalent to the last diagram in fig. 1 with $n_1 = ... = n_6 = 1$.

Let us now turn to the three-loop integrals $I^{(3)}_{n_1n_2n_3n_4n_5n_6}$. Here, we are interested in calculating only UV-divergent parts of them. It is instructive to subsequently consider three cases:

- A: All the indices $n_1, ..., n_6$ are positive.

- B: At least one of the indices is equal to zero.

- C: None of the indices vanishes, but some of them are negative.

**Case A:** From among all the three-loop integrals with six positive indices, only $I^{(3)}_{111111}$ is UV-divergent. Integrals with larger positive indices have negative degree of divergence and no subdivergences.

In order to calculate the UV-divergent part of $I^{(3)}_{111111}$ we write this integral as follows (see fig. 2):

$$I^{(3)}_{111111} \;=\; m^{-D+2}\,\pi^{-\frac{D}{2}} \int d^Dq \frac{1}{[q^2+m^2]} J^{(2)}_{11111}(q^2,m^2), \tag{13}$$

where

$$\begin{aligned} J^{(2)}_{11111}(q^2,m^2) \;&=\; m^{-2D+10}\,\pi^{-D} \times \\ \times \int d^Dq_1 d^Dq_2 &\frac{1}{[q_1^2+m^2][q_2^2+m^2][(q_1-q_2)^2+m^2][(q-q_1)^2+m^2][(q-q_2)^2+m^2]}. \end{aligned} \tag{14}$$

The latter integral is just the usual two-loop contribution to the wave function renormalization in the "$\lambda\phi^3$" theory. We show it in fig. 3. It is a finite diagram, because it has negative degree of divergence and no subdivergences. Consequently, a finite-volume integration in eqn. (13) cannot give a $1/\epsilon$ pole. Such a pole can only arise from integration over large $q^2$ in eqn. (13). Therefore, knowing the behavior of $J^{(2)}_{11111}(q^2,m^2)$ at large $q^2$ is enough to find the UV-divergent part of $I^{(3)}_{111111}$.

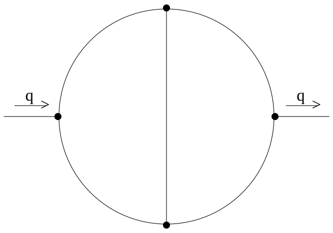


Figure 3: Graphical representation of the integral $J^{(2)}_{11111}(q^2,m^2)$.

The two-loop integral $J^{(2)}_{11111}(q^2,m^2)$ has the following expansion at large $q^2$ [3]:

$$J^{(2)}_{11111}(q^2,m^2) \underset{q^2>>m^2}{=\!=\!=} \left(\frac{m^2}{q^2}\right)^{5-D} \left[6\zeta(3) + \mathcal{O}\left(\frac{m^2}{q^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{15}$$

Inserting this result into eqn. (13) and introducing an infrared cutoff $\Lambda$ one finds

$$\begin{aligned} I^{(3)}_{111111} \;&=\; \frac{1}{\Gamma(\frac{D}{2})} \int_{\Lambda}^{\infty} d(q^2) \left(\frac{q^2}{m^2}\right)^{\frac{D}{2}-1} \frac{1}{[q^2+m^2]} J^{(2)}_{11111}(q^2,m^2) \;+\; \text{(finite terms)} \\ &=\; \left(\frac{m}{\Lambda}\right)^{6\epsilon} \frac{2\zeta(3)}{\epsilon} \;+\; \text{(finite terms)} \\ &=\; \frac{2\zeta(3)}{\epsilon} \;+\; \text{(finite terms)}. \end{aligned} \tag{16}$$

The way we have found the UV-divergent part of $I^{(3)}_{111111}$ shows the basic idea for calculating all the nontrivial three-loop integrals in the cases B and C. The UV-divergent parts of these integrals can be found by choosing some two-loop subdiagrams of the last graph in fig. 1 and considering their behavior at large external momenta. If the considered two-loop subdiagram is finite, the calculation proceeds analogously to the case of $I^{(3)}_{111111}$. If it is divergent, a subtraction of the UV divergence needs to be performed. We describe this in more detail below.

**Case B:** Now, we consider the case when at least one of the indices of $I^{(3)}_{n_1n_2n_3n_4n_5n_6}$ is equal to zero. Without loss of generality, we can assume that the vanishing index is $n_6$. This is because of the tetrahedron symmetry: The last diagram in fig. 1 has the topology of a tetrahedron. Symmetries of a tetrahedron can be described as certain permutations of its edges. Such permutations of the indices $(n_1, ..., n_6)$ leave our integral invariant.

When $n_6 = 0$, the integral can be written as

$$I^{(3)}_{n_1n_2n_3n_4n_50} = m^{-D+2n_3}\,\pi^{-\frac{D}{2}} \int d^Dq \frac{1}{[q^2+m^2]^{n_3}} G_2\left(n_1, n_5, \frac{m^2}{q^2}\right) G_2\left(n_2, n_4, \frac{m^2}{q^2}\right), \qquad (17)$$

where the one-loop integral $G_2$ is given by

$$G_2\left(k_1, k_2, \frac{m^2}{q^2}\right) = m^{-D+2k_1+2k_2}\,\pi^{-\frac{D}{2}} \int d^Dp \frac{1}{[p^2+m^2]^{k_1}[(p-q)^2+m^2]^{k_2}}. \qquad (18)$$

The diagram corresponding to eqn. (17) is shown in fig. 4.

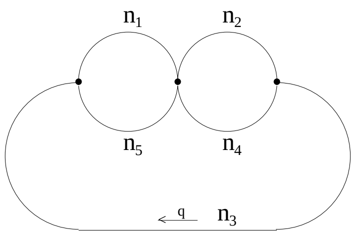


Figure 4: Graphical representation of the three-loop integral with one nonpositive index

Calculation of the integral (17) depends on what values are taken by indices of the one-loop sub-integrals denoted by $G_2$.[4] One can distinguish three situations:

- B.1: When at least one of these indices ($n_1, n_2, n_4$ or $n_5$) is nonpositive, then the three-loop integral reduces to a product of one- and two-loop tensor integrals. The

[4] However, it does not make any qualitative difference whether the index $n_3$ in the final integration is positive or not.

latter integrals can be easily reduced to scalar integrals which we are already able to calculate. Final formulae for such three-loop integrals are given in appendix B.

- B.2: When $n_1, n_2, n_4, n_5 > 0$ and both integrals $G_2$ are convergent (i.e. $n_1+n_5 > 2$ and $n_2+n_4 > 2$), the calculation proceeds analogously to the case of $I^{(3)}_{111111}$. We expand the convergent integrals $G_2$ at large $q^2$

$$G_2\left(k_1, k_2, \frac{m^2}{q^2}\right) \underset{q^2\to\infty}{=\!=\!=} \sum_{r=0}^{\infty}\left[\left(\frac{m^2}{q^2}\right)^r a(k_1,k_2,r) + \left(\frac{m^2}{q^2}\right)^{r+\epsilon} b(k_1,k_2,r)\right]. \tag{19}$$

  Only a few lowest terms in this expansion affect the pole part of the considered three-loop integral. Explicit expressions for $a(k_1,k_2,r)$ and $b(k_1,k_2,r)$ are given in appendix C.

- B.3: When $n_1, n_2, n_4, n_5 > 0$ but one or both integrals $G_2$ are divergent (i.e. $n_1 = n_5 = 1$ and/or $n_2 = n_4 = 1$), we need to split the integral $G_2$ into its pole and convergent parts

$$G_2\left(k_1, k_2, \frac{m^2}{q^2}\right) = \frac{1}{\epsilon}\delta_{1k_1}\delta_{1k_2} + G_2^{ren}\left(k_1, k_2, \frac{m^2}{q^2}\right). \tag{20}$$

  Inside the three-loop integral, the pole part of $G_2$ is multiplied by a two-loop integral. Thus, we already know how to calculate its contribution to the UV-divergence of the three-loop integral. On the other hand, the "renormalized" part of $G_2$ can be treated analogously to the case B.2, i.e. in the same way the whole $G_2$ was treated when it was convergent. Expansion of the "renormalized" integral $G_2^{ren}$ at large $q^2$ is identical as in eqn. (19), except for that $a(k_1,k_2,r)$ is replaced by

$$a^{ren}(k_1,k_2,r) = a(k_1,k_2,r) - \frac{1}{\epsilon}\delta_{1k_1}\delta_{1k_2}\delta_{0r}. \tag{21}$$

  One should not naively expect that $a^{ren}(k_1,k_2,r)$ contain no poles in $\epsilon$. Actually, they do contain simple poles which cancel with the poles of $b(k_1,k_2,r)$ in the expression for $G_2^{ren}$.

**Case C:** Now, we consider integrals with some negative indices but with none of them equal to zero. Using the tetrahedron symmetry, we can assume without loss of generality that one of the negative indices is $n_6$. Then, we consider two distinct situations:

- C.1: When any of the indices $n_1, n_2, n_4$ or $n_5$ is negative, the three-loop integral reduces to products of one- and two-loop integrals, similarly to the case when $n_6 = 0$.

The explicit formulae given in appendix B apply both when $n_6$ vanishes and when it is negative.

- C.2: When all the remaining indices are positive or the only other negative index is $n_3$, we can still represent the considered three-loop integral by the diagram shown in fig. 4. However, instead of the scalar one-loop integrals $G_2$, we encounter tensor one-loop integrals. This does not lead to any real difficulty, because we are able to reduce tensor integrals to scalar ones. Nevertheless, the amount of necessary algebra can be drastically reduced when one makes use of certain tensor identities discussed in appendix D.

In the above considerations, we have described a complete algorithm for calculating pole parts of the integrals defined in eqns. (6)–(8). However, obtaining final formulae for nontrivial three-loop integrals in the cases B.2, B.3 and C.2 requires discussing a few more subtle points in their evaluation. This is what the next section is devoted to.

## 4. More on nontrivial three-loop integrals.

Let us first derive our final expression for the three-loop integrals in the cases B.2 and B.3. In both these cases, the considered three-loop integral can be written as

$$
\begin{aligned}
I^{(3)}_{n_1 n_2 n_3 n_4 n_5 0} &= m^{-D+2n_3}\,\pi^{-\frac{D}{2}} \int d^D q \frac{1}{[q^2+m^2]^{n_3}} G_2^{ren}\left(n_1, n_5, \frac{m^2}{q^2}\right) G_2^{ren}\left(n_2, n_4, \frac{m^2}{q^2}\right) \\
&\quad + \frac{1}{\epsilon}\delta_{1n_1}\delta_{1n_5} I^{(2)}_{n_2 n_4 n_3} + \frac{1}{\epsilon}\delta_{1n_2}\delta_{1n_4} I^{(2)}_{n_1 n_5 n_3} - \frac{1}{\epsilon^2}\delta_{1n_1}\delta_{1n_2}\delta_{1n_4}\delta_{1n_5} I^{(1)}_{n_3} = \\
&= (\text{finite terms}) + \frac{1}{\Gamma(\frac{D}{2})}\int_0^{\frac{m^2}{\Lambda^2}} d\left(\frac{m^2}{q^2}\right)\left(1+\frac{m^2}{q^2}\right)^{-n_3} \sum_{r_1,r_2=0}^{\infty}\left(\frac{m^2}{q^2}\right)^{n_3+r_1+r_2-3} \times \\
&\quad \times \left\{ a^{ren}(n_1,n_5,r_1)\, a^{ren}(n_2,n_4,r_2)\left(\frac{m^2}{q^2}\right)^{\epsilon} + b(n_1,n_5,r_1)\, b(n_2,n_4,r_2)\left(\frac{m^2}{q^2}\right)^{3\epsilon} \right. \\
&\quad \left. + \left[a^{ren}(n_1,n_5,r_1)\, b(n_2,n_4,r_2) + b(n_1,n_5,r_1)\, a^{ren}(n_2,n_4,r_2)\right]\left(\frac{m^2}{q^2}\right)^{2\epsilon} \right\} \\
&\quad + \frac{1}{\epsilon}\delta_{1n_1}\delta_{1n_5} I^{(2)}_{n_2 n_4 n_3} + \frac{1}{\epsilon}\delta_{1n_2}\delta_{1n_4} I^{(2)}_{n_1 n_5 n_3} - \frac{1}{\epsilon^2}\delta_{1n_1}\delta_{1n_2}\delta_{1n_4}\delta_{1n_5} I^{(1)}_{n_3}. \qquad (22)
\end{aligned}
$$

Similarly to eqn. (16), an arbitrary infrared cutoff $\Lambda$ has been introduced here. Assuming that $\Lambda^2 \geq m^2$, we can expand in eqn. (22)

$$
\left(1+\frac{m^2}{q^2}\right)^{-n_3} = \sum_{k=0}^{\infty} \frac{(-n_3)(-n_3-1)...(-n_3-k+1)}{k!}\left(\frac{m^2}{q^2}\right)^k \equiv \sum_{k=0}^{\infty}\binom{-n_3}{k}\left(\frac{m^2}{q^2}\right)^k. \qquad (23)
$$

After performing trivial integrations, we arrive at the following result:

$$
\begin{aligned}
I^{(3)}_{n_1n_2n_3n_4n_50} &= (\text{finite terms}) + \frac{1}{\Gamma(\frac{D}{2})}\sum_{r_1,r_2,k=0}^{\infty}\left(\begin{array}{c} -n_3 \\ k \end{array}\right)\left(\frac{m^2}{\Lambda^2}\right)^{n_3-2+r_1+r_2+k} \times \\
&\times \left\{ \frac{a^{ren}(n_1,n_5,r_1)a^{ren}(n_2,n_4,r_2)}{n_3-2+r_1+r_2+k+\epsilon}\left(\frac{m^2}{\Lambda^2}\right)^{\epsilon} + \frac{b(n_1,n_5,r_1)b(n_2,n_4,r_2)}{n_3-2+r_1+r_2+k+3\epsilon}\left(\frac{m^2}{\Lambda^2}\right)^{3\epsilon} \right. \\
&\quad \left. + \frac{a^{ren}(n_1,n_5,r_1)b(n_2,n_4,r_2)+b(n_1,n_5,r_1)a^{ren}(n_2,n_4,r_2)}{n_3-2+r_1+r_2+k+2\epsilon}\left(\frac{m^2}{\Lambda^2}\right)^{2\epsilon} \right\} \\
&+ \frac{1}{\epsilon}\delta_{1n_1}\delta_{1n_5}I^{(2)}_{n_2n_4n_3} + \frac{1}{\epsilon}\delta_{1n_2}\delta_{1n_4}I^{(2)}_{n_1n_5n_3} - \frac{1}{\epsilon^2}\delta_{1n_1}\delta_{1n_2}\delta_{1n_4}\delta_{1n_5}I^{(1)}_{n_3}. \qquad (24)
\end{aligned}
$$

The curly bracket in the above equation contains no $1/\epsilon$ poles unless $n_3-2+r_1+r_2+k=0$. Verifying this requires a short calculation, because $a^{ren}(k_1,k_2,r)$ and $b(k_1,k_2,r)$ do contain simple poles in $\epsilon$. Thus, for $n_3-2+r_1+r_2+k \neq 0$, one needs to expand the denominators to $\mathcal{O}(\epsilon)$ and check that the potential $1/\epsilon$ contributions to $I^{(3)}$ "miraculously" sum up to zero, due to

$$
a^{ren}(k_1,k_2,r) + b(k_1,k_2,r) = \mathcal{O}(1). \qquad (25)
$$

Our final expression for $I^{(3)}_{n_1n_2n_3n_4n_50}$ in the cases B.2 and B.3 is thus given by a finite sum (from now on we set $\Lambda = m$ for simplicity[5] ):

$$
\begin{aligned}
I^{(3)}_{n_1n_2n_3n_4n_50} &= (\text{finite terms}) + \frac{1}{\epsilon\Gamma(2-\epsilon)}\sum_{r_1=0}^{2-n_3}\sum_{r_2=0}^{2-n_3-r_1}\left(\begin{array}{c} -n_3 \\ 2-n_3-r_1-r_2 \end{array}\right) \times \\
&\times \left\{ a^{ren}(n_1,n_5,r_1)a^{ren}(n_2,n_4,r_2) + \frac{1}{3}b(n_1,n_5,r_1)b(n_2,n_4,r_2) \right. \\
&\quad \left. + \frac{1}{2}\left[a^{ren}(n_1,n_5,r_1)b(n_2,n_4,r_2) + b(n_1,n_5,r_1)a^{ren}(n_2,n_4,r_2)\right] \right\} \\
&+ \frac{1}{\epsilon}\delta_{1n_1}\delta_{1n_5}I^{(2)}_{n_2n_4n_3} + \frac{1}{\epsilon}\delta_{1n_2}\delta_{1n_4}I^{(2)}_{n_1n_5n_3} - \frac{1}{\epsilon^2}\delta_{1n_1}\delta_{1n_2}\delta_{1n_4}\delta_{1n_5}I^{(1)}_{n_3}. \qquad (26)
\end{aligned}
$$

Let us now turn to the most complicated case C.2. In this case, the indices $n_1, n_2, n_4$ and $n_5$ are positive, while the index $n_6$ is negative. Using the tensor identities given in the end of appendix D, we express $[(q_1-q_2)^2+m^2]^{-n_6}$ in terms of symmetric and traceless tensors

$$
\begin{aligned}
[(q_1-q_2)^2+m^2]^{-n_6} &= \sum_{k=0}^{-n_6}\sum_{i=0}^{k}\sum_{\rho=0}^{[i/2]}\sum_{l_1=0}^{\rho}\sum_{l_2=0}^{k-i+\rho}\left(\begin{array}{c} -n_6 \\ k \end{array}\right)\left(\begin{array}{c} k \\ i \end{array}\right)\left(\begin{array}{c} \rho \\ l_1 \end{array}\right)\left(\begin{array}{c} k-i+\rho \\ l_2 \end{array}\right) \times \\
&\times \frac{i!2^i(-1)^{i+l_1+l_2}(m^2)^{l_1+l_2}}{4^\rho \rho!(i-2\rho)!(i+2-2\rho-\epsilon)_\rho}(q_1^2+m^2)^{-n_6-k+\rho-l_1}(q_2^2+m^2)^{k-i+\rho-l_2}(q_1\cdot q_2)^{(i-2\rho)}, \qquad (27)
\end{aligned}
$$

[5] One could keep $\Lambda$ arbitrary and verify that the pole part of $I^{(3)}$ is independent of this parameter. This can serve as a useful cross-check against misprints in the explicit expressions for $a(k_1,k_2,r)$ and $b(k_1,k_2,r)$ in appendix C.

where $(x)_n$ denotes the Pochhammer symbol

$$(x)_n = x(x+1)(x+2)...(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{28}$$

Consequently, we can write

$$\begin{aligned} I^{(3)}_{n_1n_2n_3n_4n_5n_6} \underset{n_6<0}{=\!=\!=} & \sum_{k=0}^{-n_6}\sum_{i=0}^{k}\sum_{\rho=0}^{[i/2]}\sum_{l_1=0}^{\rho}\sum_{l_2=0}^{k-i+\rho}\binom{-n_6}{k}\binom{k}{i}\binom{\rho}{l_1}\binom{k-i+\rho}{l_2}\times \\ & \times\frac{i!2^{i-2\rho}(-1)^{i+l_1+l_2}}{\rho!(i-2\rho)!(i+2-2\rho-\epsilon)_\rho} I^{(3)(i-2\rho)}_{(n_1+n_6+k-\rho+l_1)(n_2-k+i-\rho+l_2)n_3n_4n_50} \end{aligned} \tag{29}$$

where

$$\begin{aligned} I^{(3)(n)}_{n_1n_2n_3n_4n_50} &= m^{-3D-2n+2\Sigma n_i}\,\pi^{-\frac{3D}{2}}\int d^Dq_1\,d^Dq_2\,d^Dq_3\times \\ &\times\frac{(q_1\cdot q_2)^{(n)}}{[q_1^2+m^2]^{n_1}[q_2^2+m^2]^{n_2}[q_3^2+m^2]^{n_3}[(q_2-q_3)^2+m^2]^{n_4}[(q_3-q_1)^2+m^2]^{n_5}}. \end{aligned} \tag{30}$$

The above integral is a generalization of $I^{(3)(0)}_{n_1n_2n_3n_4n_50} \equiv I^{(3)}_{n_1n_2n_3n_4n_50}$ considered in the case B. When $n_1, n_2, n_4$ or $n_5$ is nonpositive, $I^{(3)(n)}_{n_1n_2n_3n_4n_50}$ is equal to a linear combination of reducible integrals considered in the cases B.1 and C.1. The explicit form of this linear combination is given in the end of appendix B. On the other hand, when $n_1, n_2, n_4$ and $n_5$ are all positive, the calculation of $I^{(3)(n)}_{n_1n_2n_3n_4n_50}$ proceeds analogously to the cases B.2 and B.3. However, instead of the scalar integrals $G_2$, we encounter tensor one-loop integrals with totally symmetric and traceless tensors in their numerators. Such one-loop integrals are in one-to-one correspondence with scalar one-loop integrals in larger number of dimensions. The appropriate relation is given in appendix D. Using this relation, one finds the necessary generalization of eqn. (17)

$$\begin{aligned} I^{(3)(n)}_{n_1n_2n_3n_4n_50} &= (n_4)_n(n_5)_n m^{-4+2\epsilon-2n+2n_3}\,\pi^{-2+\epsilon}\int d^{4-2\epsilon}q\frac{(q\cdot q)^{(n)}}{[q^2+m^2]^{n_3}}\times \\ &\times\; G_2\left(n_1, n_5+n, \frac{m^2}{q^2}\right)\bigg|_{D=4+2n-2\epsilon} G_2\left(n_2, n_4+n, \frac{m^2}{q^2}\right)\bigg|_{D=4+2n-2\epsilon}, \end{aligned} \tag{31}$$

where $(q\cdot q)^{(n)}$ can be expressed back in terms of $q^2$

$$(q\cdot q)^{(n)} = \frac{(2-2\epsilon)_n}{2^n(1-\epsilon)_n}(q^2)^n. \tag{32}$$

Similarly to eqn. (20), we split the higher-dimensional $G_2$ into its pole and convergent parts

$$G_2\left(k_1, k_2+n, \frac{m^2}{q^2}\right)\bigg|_{D=4+2n-2\epsilon} = \frac{1}{\epsilon(n+1)!}\delta_{1k_1}\delta_{1k_2} + G_2^{ren}\left(k_1, k_2+n, \frac{m^2}{q^2}\right)\bigg|_{D=4+2n-2\epsilon}. \tag{33}$$

Next, we expand the convergent part at large $q^2$, as in eqn. (19)

$$G_2^{ren}\left(k_1, k_2+n, \frac{m^2}{q^2}\right)\Bigg|_{D=4+2n-2\epsilon} \underset{q^2\to\infty}{=\!=}$$

$$\underset{q^2\to\infty}{=\!=} \sum_{r=0}^{\infty}\left[\left(\frac{m^2}{q^2}\right)^r A^{ren}(k_1,k_2+n,2+n,r) + \left(\frac{m^2}{q^2}\right)^{r+\epsilon} B(k_1,k_2+n,2+n,r)\right], \qquad (34)$$

where

$$A^{ren}(k_1,k_2+n,2+n,r) = A(k_1,k_2+n,2+n,r) - \frac{1}{\epsilon(n+1)!}\delta_{1k_1}\delta_{1k_2}\delta_{0r} \qquad (35)$$

The coefficients $A(k_1,k_2,\omega,r)$ and $B(k_1,k_2,\omega,r)$ are given explicitly in appendix C.

At this point, we are ready to write down the desired generalization of eqn. (26)

$$\begin{aligned}
I^{(3)(n)}_{n_1n_2n_3n_4n_50} \;=\; & \text{(finite terms)} \;+ \\
&+ \frac{(n_4)_n(n_5)_n(2-2\epsilon)_n}{2^n(1-\epsilon)_n}\frac{1}{\epsilon\Gamma(2-\epsilon)}\sum_{r_1=0}^{2+n-n_3}\sum_{r_2=0}^{2+n-n_3-r_1}\binom{-n_3}{2+n-n_3-r_1-r_2}\times \\
&\qquad \times\{A^{ren}(n_1,n_5+n,2+n,r_1)A^{ren}(n_2,n_4+n,2+n,r_2) \\
&\qquad\quad +\frac{1}{3}B(n_1,n_5+n,2+n,r_1)B(n_2,n_4+n,2+n,r_2) \\
&\qquad\quad +\frac{1}{2}\left[A^{ren}(n_1,n_5+n,2+n,r_1)B(n_2,n_4+n,2+n,r_2)\right. \\
&\qquad\quad \left.+B(n_1,n_5+n,2+n,r_1)A^{ren}(n_2,n_4+n,2+n,r_2)\right]\} \\
&+\frac{n!}{\epsilon\, 2^n(n+1)}\sum_{\rho=0}^{[n/2]}\sum_{i=0}^{\rho}\sum_{j=0}^{n-2\rho}\sum_{l=0}^{j}\sum_{k=0}^{n-\rho-j}\binom{\rho}{i}\binom{n-2\rho}{j}\binom{j}{l}\binom{n-\rho-j}{k}\times \\
&\qquad \times\frac{(-1)^{n+\rho+i+k+l}}{\rho!(n-2\rho)!(n+1-\rho-\epsilon)_\rho}\left[\delta_{1n_1}\delta_{1n_5}I^{(2)}_{(n_2-i-l)(n_4-j+l)(n_3-k)}\right. \\
&\qquad\qquad \left.+\delta_{1n_2}\delta_{1n_4}I^{(2)}_{(n_1-i-l)(n_5-j+l)(n_3-k)}\right] \\
&-\delta_{1n_1}\delta_{1n_2}\delta_{1n_4}\delta_{1n_5}\frac{(2-2\epsilon)_n(2-\epsilon)_n}{\epsilon^2\, 2^n(n+1)^2(1-\epsilon)_n}\frac{\Gamma(n_3-n-2+\epsilon)}{\Gamma(n_3)}.
\end{aligned} \qquad (36)$$

The above equation is the main result of the present paper. It gives us pole parts of all the nontrivial scalar three-loop integrals $I^{(3)}$, i.e. those which do not reduce to products of lower-loop integrals. When $n=0$, it reduces to eqn. (26).

## 5. From renormalization constants to $\beta$-functions and anomalous dimensions.

In the preceding sections, we have described an algorithm for calculating pole parts of Feynman diagrams. Using our formulae, one can find all the MS-scheme renormalization constants in a given theory, up to three loops. In the present short section, we give two examples of relations between three-loop renormalization constants and beta functions or anomalous dimensions.

Here, we depart from the MS scheme and assume that the renormalization constants (calculated in the framework of dimensional regularization) can contain arbitrary finite terms. However, we assume that these finite terms are renormalization-scale independent.

For instance, let us consider renormalization of the gauge coupling $g$ in some Yang-Mills theory

$$g^{BARE} = \mu^{\epsilon}\, Z_g\, g, \tag{37}$$

where $\mu$ is the renormalization scale. The renormalization constant $Z_g$ has the following expansion in powers of the renormalized coupling $g$:

$$Z_g = 1 + g^2\left(\kappa^{01} + \frac{\kappa^{11}}{\epsilon}\right) + g^4\left(\kappa^{02} + \frac{\kappa^{12}}{\epsilon} + \frac{\kappa^{22}}{\epsilon^2}\right) + g^6\left(\kappa^{03} + \frac{\kappa^{13}}{\epsilon} + \frac{\kappa^{23}}{\epsilon^2} + \frac{\kappa^{33}}{\epsilon^3}\right) + \ldots \tag{38}$$

Some coefficients in this expansion are given in terms of the others, which follows from locality of UV-divergences

$$\begin{aligned}
\kappa^{22} &= \frac{3}{2}(\kappa^{11})^2 \\
\kappa^{33} &= \frac{5}{2}(\kappa^{11})^3 \\
\kappa^{23} &= \frac{11}{3}\kappa^{11}\kappa^{12} - \frac{7}{2}\kappa^{01}(\kappa^{11})^2.
\end{aligned} \tag{39}$$

From scale-independence of $g^{BARE}$ one can derive the following expression for the $\beta$-function in terms of $\kappa^{ij}$:

$$\beta(g) \equiv \mu\frac{dg}{d\mu} = 2\kappa^{11}g^3 + \left[4\kappa^{12} - 12\kappa^{01}\kappa^{11}\right]g^5 + \left[6\kappa^{13} - 22\kappa^{11}\kappa^{02} - 22\kappa^{01}\kappa^{12} + 54\kappa^{11}(\kappa^{01})^2\right]g^7 + \ldots \tag{40}$$

As another example, let us discuss the anomalous dimension matrix of a set of (possibly dimensionful) couplings $C_i$ which linearly mix under renormalization

$$C_j^{BARE} = \sum_i C_i Z_{ij} = \left(\overline{C}^T \hat{Z}\right)_j. \tag{41}$$

Let us assume, that the renormalization constant matrix $\hat{Z}$ depends on a single gauge coupling $g$. Then it reads

$$\hat{Z} = 1 + g^2 \left( \hat{a}^{01} + \frac{\hat{a}^{11}}{\epsilon} \right) + g^4 \left( \hat{a}^{02} + \frac{\hat{a}^{12}}{\epsilon} + \frac{\hat{a}^{22}}{\epsilon^2} \right) + g^6 \left( \hat{a}^{03} + \frac{\hat{a}^{13}}{\epsilon} + \frac{\hat{a}^{23}}{\epsilon^2} + \frac{\hat{a}^{33}}{\epsilon^3} \right) + ... \quad (42)$$

Some coefficients in the above expansion are given in terms of the others, which follows from locality of UV-divergences

$$\begin{aligned} \hat{a}^{22} &= \frac{1}{2}(\hat{a}^{11})^2 + \kappa^{11}\hat{a}^{11} \\ \hat{a}^{33} &= \frac{1}{6}(\hat{a}^{11})^3 + \kappa^{11}(\hat{a}^{11})^2 + \frac{4}{3}(\kappa^{11})^2\hat{a}^{11} \\ \hat{a}^{23} &= \frac{1}{3}\hat{a}^{11}\hat{a}^{12} + \frac{2}{3}\hat{a}^{12}\hat{a}^{11} - \frac{1}{6}\hat{a}^{01}(\hat{a}^{11})^2 - \frac{1}{3}\hat{a}^{11}\hat{a}^{01}\hat{a}^{11} - \frac{1}{3}\kappa^{11}\hat{a}^{01}\hat{a}^{11} + \frac{4}{3}\kappa^{11}\hat{a}^{12} + \left(\frac{4}{3}\kappa^{12} - \frac{8}{3}\kappa^{01}\kappa^{11}\right)\hat{a}^{11}. \end{aligned} \quad (43)$$

Scale-independence of $\overline{C}^{BARE}$ implies that the renormalized couplings $C_i$ satisfy the following renormalization group equations

$$\mu\frac{d}{d\mu}\overline{C} = \hat{\gamma}^T\overline{C}, \quad (44)$$

where the anomalous dimension matrix $\hat{\gamma}$ has the following expansion in powers of $g$

$$\begin{aligned} \hat{\gamma} = \; & 2\hat{a}^{11}g^2 + g^4\left[4\hat{a}^{12} - 2\hat{a}^{01}\hat{a}^{11} - 2\hat{a}^{11}\hat{a}^{01} - 4\kappa^{11}\hat{a}^{01} - 4\kappa^{01}\hat{a}^{11}\right] + g^6\left[6\hat{a}^{13} - 4\hat{a}^{12}\hat{a}^{01} - 2\hat{a}^{01}\hat{a}^{12}\right. \\ & -4\hat{a}^{02}\hat{a}^{11} - 2\hat{a}^{11}\hat{a}^{02} + 2\hat{a}^{01}\hat{a}^{11}\hat{a}^{01} + 2\hat{a}^{11}(\hat{a}^{01})^2 + 2(\hat{a}^{01})^2\hat{a}^{11} + 4\kappa^{11}(\hat{a}^{01})^2 - 8\kappa^{11}\hat{a}^{02} \\ & \left. +4\kappa^{01}\hat{a}^{11}\hat{a}^{01} + 4\kappa^{01}\hat{a}^{01}\hat{a}^{11} - 8\kappa^{01}\hat{a}^{12} - 8\kappa^{12}\hat{a}^{01} - 8\kappa^{02}\hat{a}^{11} + 24\kappa^{01}\kappa^{11}\hat{a}^{01} + 12(\kappa^{01})^2\hat{a}^{11}\right] + ... \end{aligned} \quad (45)$$

In the MS scheme, equations (40) and (45) become much simpler

$$\beta(g) = 2\kappa^{11}g^3 + 4\kappa^{12}g^5 + 6\kappa^{13}g^7 + ... \quad (46)$$

$$\hat{\gamma} = 2\hat{a}^{11}g^2 + 4\hat{a}^{12}g^4 + 6\hat{a}^{13}g^6 + ... \quad (47)$$

However, using the pure MS scheme may not be possible in some effective theories where so-called "evanescent operators" arise in dimensional regularization. This is why the more general relations (40) and (45) have been presented here.

## 6. Summary.

We have described an algorithm for calculating UV-divergent parts of arbitrary Feynman diagrams. A common mass parameter has been used to perform the infrared rearrangement. Explicit formulae for all the necessary scalar integrals up to three loops have been given.

The main idea in calculating nontrivial three-loop integrals was considering some of their two-loop subintegrals and expanding them at large external momenta. In the end, some details have been given on relations between UV-divergences and $\beta$-functions or anomalous dimensions.

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## Appendix A

This appendix is devoted to reduction of tensor integrals to scalar ones. We are interested in Feynman integrands depending only on loop momenta and the artificial mass parameter $m^2$. The integrals arising at one, two and three loops have the following form (in the euclidean metric):

$$(T^{(1)k}_{n})_{\mu_1 \ldots \mu_k} = m^{-D-k+2n} \, \pi^{-\frac{D}{2}} \int \frac{d^D p \quad p_{\mu_1} \ldots p_{\mu_k}}{[p^2+m^2]^n}, \tag{48}$$

$$(T^{(2)kl}_{n_1 n_2 n_3})_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_l} = m^{-2D-k-l+2\sum n_i} \, \pi^{-D} \int \frac{d^D p \, d^D q \quad p_{\mu_1} \ldots p_{\mu_k} \, q_{\nu_1} \ldots q_{\nu_l}}{[p^2+m^2]^{n_1} [q^2+m^2]^{n_2} [(p-q)^2+m^2]^{n_3}}, \tag{49}$$

$$(T^{(3)klm}_{n_1 n_2 n_3 n_4 n_5 n_6})_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_l \rho_1 \ldots \rho_m} = m^{-3D-k-l-m+2\sum n_i} \, \pi^{-\frac{3D}{2}} \times$$

$$\times \int \frac{d^D p \, d^D q \, d^D r \quad p_{\mu_1} \ldots p_{\mu_k} \, q_{\nu_1} \ldots q_{\nu_l} \, r_{\rho_1} \ldots r_{\rho_m}}{[p^2+m^2]^{n_1} [q^2+m^2]^{n_2} [r^2+m^2]^{n_3} [(q-r)^2+m^2]^{n_4} [(r-p)^2+m^2]^{n_5} [(p-q)^2+m^2]^{n_6}}. \tag{50}$$

Such integrals are proportional to linear combinations of products of metric tensors. For instance,

$$(T^{(2)22}_{n_1 n_2 n_3})_{\mu\nu\rho\sigma} = F_1\, g_{\mu\nu} g_{\rho\sigma} + F_2\, (g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}). \tag{51}$$

The tensors $g_{\mu\rho} g_{\nu\sigma}$ and $g_{\mu\sigma} g_{\nu\rho}$ are multiplied by the same coefficient $F_2$ in the above equation, due to an obvious symmetry. In a computer algebra code, such a symmetry can be verified by checking that contractions of the l.h.s of eqn. (51) with $g_{\mu\rho} g_{\nu\sigma}$ and $g_{\mu\sigma} g_{\nu\rho}$ are identical.

The equations for the coefficients $F_1$ and $F_2$ are found by contracting the tensor integral with $g_{\mu\nu} g_{\rho\sigma}$ and $g_{\mu\rho} g_{\nu\sigma}$

$$\left\{ \begin{array}{rcl} D^2 F_1 + 2 D F_2 & = & X_1 \\ D F_1 + D(D+1) F_2 & = & X_2 \end{array} \right. \tag{52}$$

where

$$X_1 = m^{-2D-4+2\sum n_i}\, \pi^{-D} \int \frac{d^D p\, d^D q \quad p^2 q^2}{[p^2+m^2]^{n_1} [q^2+m^2]^{n_2} [(p-q)^2+m^2]^{n_3}}, \tag{53}$$

$$X_2 = m^{-2D-4+2\sum n_i}\, \pi^{-D} \int \frac{d^D p\, d^D q \quad (p\cdot q)^2}{[p^2+m^2]^{n_1} [q^2+m^2]^{n_2} [(p-q)^2+m^2]^{n_3}}. \tag{54}$$

Consequently,

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} D^2 & 2D \\ D & D(D+1) \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}. \tag{55}$$

The above matrix inversion is most easily done perturbatively in $\epsilon$, after substituting $D = 4 - 2\epsilon$. This makes the computer program much faster, which is important for more complicated tensor integrals where larger matrices need to be inverted.

The integrals like $X_1$ and $X_2$ are easily reduced to the standard scalar integrals (7), with help of the identities

$$p^2 = (p^2+m^2) - m^2, \tag{56}$$

$$q^2 = (q^2+m^2) - m^2, \tag{57}$$

$$p\cdot q = \frac{1}{2}\{(p^2+m^2) + (q^2+m^2) - [(p-q)^2+m^2] - m^2\}. \tag{58}$$

Specific values for the indices $n_i$ can be substituted only after all these operations are performed.

## Appendix B

In this appendix, we give explicit formulae for the trivial integrals, i.e. for the two-loop integrals which reduce to products of one-loop ones, and for the three-loop integrals which reduce to products of one- and two-loop ones.

When at least one of the indices of the two-loop integral $I^{(2)}_{n_1n_2n_3}$ is nonpositive, the integral reduces to a product of tensor one-loop integrals. However, a simple expression for such an integral can be also obtained from a general formula for a two-loop integral with one massless and two massive lines [7]

$$
\begin{aligned}
I^{(2)}_{n_1n_2n_3} \underset{n_3\le 0}{=\!=\!=} & \sum_{k=0}^{-n_3} \binom{-n_3}{k} m^{-2D+2(n_1+n_2+k)}\pi^{-D} \int \frac{d^Dq_1\, d^Dq_2}{[q_1^2+m^2]^{n_1}[q_2^2+m^2]^{n_2}[(q_1-q_2)^2]^{-k}} \\
= & \sum_{k=0}^{-n_3} \binom{-n_3}{k} \frac{\Gamma(\frac{D}{2}+k)\Gamma(n_1-k-\frac{D}{2})\Gamma(n_2-k-\frac{D}{2})\Gamma(n_1+n_2-k-D)}{\Gamma(n_1)\Gamma(n_2)\Gamma(\frac{D}{2})\Gamma(n_1+n_2-2k-D)} \qquad (59)
\end{aligned}
$$

In the above equation, we have assumed that the nonpositive index is $n_3$. This could have been done without loss of generality, because $I^{(2)}_{n_1n_2n_3}$ is totally symmetric under permutations of its indices. Equation (59) implies that $I^{(2)}_{n_1n_2n_3}$ vanishes when more than one of its indices is nonpositive.

Let us now turn to the three-loop integrals $I^{(3)}_{n_1n_2n_3n_4n_5n_6}$ considered in the cases B.1 and C.1 in the main text. They have nonpositive index $n_6$ and, in addition, there is one more nonpositive index among $n_1, n_2, n_4$ and $n_5$. Symmetries of the diagram shown in fig. 4 allow to assume without loss of generality that the other nonpositive index is $n_5$. The remaining indices can be arbitrary integers. In such a case, the three loop integral is expressible in terms of tensor one- and two-loop integrals as follows:

$$
\begin{aligned}
I^{(3)}_{n_1n_2n_3n_4n_5n_6} = & \sum_{j_5=0}^{-n_5}\sum_{j_6=0}^{-n_6}\sum_{i_5=0}^{j_5}\sum_{i_6=0}^{j_6}\sum_{k_5=0}^{-n_5-j_5}\sum_{k_6=0}^{-n_6-j_6} (-1)^{j_5+j_6}\, 2^{i_5+i_6} \binom{-n_5}{j_5}\binom{-n_6}{j_6}\binom{j_5}{i_5}\binom{j_6}{i_6} \times \\
& \times \binom{-n_5-j_5}{k_5}\binom{-n_6-j_6}{k_6} \left(T^{(1)(i_5+i_6)}_{n_1-k_5-k_6}\right)^{\mu_1\ldots\mu_{i_5+i_6}} \left(T^{(2)i_6i_5}_{(n_2+n_6+j_6+k_6)(n_3+n_5+j_5+k_5)n_4}\right)_{\mu_1\ldots\mu_{i_5+i_6}} \qquad (60)
\end{aligned}
$$

The tensor integrals appearing in the above equation have been defined in appendix A.

In the end, we consider the integral $I^{(3)(n)}_{n_1n_2n_3n_4n_50}$ defined in eqn. (30) in the reducible case, i.e. when $n_1, n_2, n_4$ or $n_5$ is nonpositive. In such a case, we can calculate $I^{(3)(n)}_{n_1n_2n_3n_4n_50}$ by expressing it as a linear combination of the integrals considered in the previous paragraph

$$
\begin{aligned}
I^{(3)(n)}_{n_1n_2n_3n_4n_50} = & \sum_{\rho=0}^{[n/2]}\sum_{k=0}^{n-2\rho}\sum_{i=0}^{k}\sum_{l_1=0}^{\rho}\sum_{l_2=0}^{k-i+\rho} \binom{n-2\rho}{k}\binom{k}{i}\binom{\rho}{l_1}\binom{k-i+\rho}{l_2} \times \\
& \times \frac{(-1)^{\rho+n+i+l_1+l_2} n!}{(n+1-\rho-\epsilon)_\rho 2^n \rho!(n-2\rho)!}\, I^{(3)}_{(n_1-i-l_1)(n_2-l_2)n_3n_4n_5(k+2\rho-n)}. \qquad (61)
\end{aligned}
$$

In some of the integrals $I^{(3)}$ on the r.h.s. of the above equation, we may need to permute the first five indices using the tetrahedron symmetry. The fifth index must become nonpositive before eqn. (60) is applied.

**Appendix C**

Here, we give explicit formulae for the coefficients $a(k_1,k_2,r)$ and $b(k_1,k_2,r)$ in the expansion (19) of $G_2$ at large $q^2$. In section 4, we also need their generalizations to $D=2\omega-2\epsilon$ dimensional space, with arbitrary $\omega$. Thus, we write

$$\begin{aligned} a(k_1,k_2,r) &= A(k_1,k_2,\omega=2,r) && (62)\\ b(k_1,k_2,r) &= B(k_1,k_2,\omega=2,r). && (63)\end{aligned}$$

The above quantities are symmetric with respect to their first two arguments. Moreover, we are only interested in positive $k_1$ and $k_2$ in our application to three-loop integrals. Consequently, knowing $A$ and $B$ for $1\le k_1\le k_2$ is everything we need here.

Using eqns. (18) and (A.1) of ref. [22],[6] one finds (for $1\le k_1\le k_2$)

$$A(k_1,k_2,\omega,r)=\begin{cases} 0 & \text{when } r<k_1 \text{ or } \frac{k_1+k_2}{2}\le r<k_2 \\ \frac{(-1)^{r-k_1}(r-1)!(k_2-r-1)!\Gamma(k_1+k_2-r-\omega+\epsilon)}{(k_1-1)!(k_2-1)!(r-k_1)!(k_1+k_2-2r-1)!} & \text{when } k_1\le r<\frac{k_1+k_2}{2} \\ \frac{2(r-1)!(2r-k_1-k_2)!\Gamma(k_1+k_2-r-\omega+\epsilon)}{(k_1-1)!(k_2-1)!(r-k_1)!(r-k_2)!} & \text{when } r\ge k_2 \end{cases} \qquad (64)$$

$$B(k_1,k_2,\omega,r)=\begin{cases} 0 & \text{when } r<k_1+k_2-\omega \\ \frac{(-1)^{r-k_1-k_2+\omega}\Gamma(k_1-r-\epsilon)\Gamma(k_2-r-\epsilon)\Gamma(r+\epsilon)}{(k_1-1)!(k_2-1)!(r-k_1-k_2+\omega)!\Gamma(k_1+k_2-2r-2\epsilon)} & \text{when } r\ge k_1+k_2-\omega \end{cases} \qquad (65)$$

The coefficients $A$ and $B$ often contain simple poles in $\epsilon$. For convergent integrals, these poles are usually also present, but they cancel out in the final expression for $G_2$. Thus, even when both $G_2$ in eqn. (17) are convergent, one needs to carefully keep track of the $\mathcal{O}(\epsilon)$ parts in $A$ and $B$.

**Appendix D**

Here, we present a useful relation between tensor and scalar one-loop integrals in different numbers of dimensions. Let $p^{(\alpha_1\dots\alpha_n)}$ denote the only symmetric and traceless[7] rank $n$ tensor which can be formed from a D-vector $p$ (see eg. [23])

$$p^{(\mu_1\dots\mu_n)}=\hat{S}\sum_{\rho=0}^{[n/2]}\frac{(-1)^\rho n!}{4^\rho\rho!(n-2\rho)!(n+1-\rho-\epsilon)_\rho}g^{\mu_1\mu_2}\dots g^{\mu_{2\rho-1}\mu_{2\rho}}p^{2\rho}p^{\mu_{2\rho+1}}\dots p^{\mu_n}. \qquad (66)$$

Here, $\hat{S}$ stands for the operator which symmetrizes with respect to all $n$ indices and multiplies the result by $1/n!$.

[6] Elementary identities $\Gamma(z)\Gamma(z+1/2)=2^{1-2z}\Gamma(2z)$ and $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ have been used to simplify these expressions.

[7] Tracelessness of a rank $n$ tensor means vanishing of any of its contractions with $g_{\mu\nu}$.

The tensors $p^{(\alpha_1\ldots\alpha_n)}$ occur in a useful relation between one-loop tensor and scalar integrals in different numbers of dimensions [24]

$$\begin{aligned}
m^{-D+2k_1+2k_2} \quad & \pi^{-D/2}\int d^Dp\frac{p^{(\alpha_1\ldots\alpha_n)}}{[p^2+m^2]^{k_1}[(p-q)^2+m^2]^{k_2}} = \\
= \quad & (k_2)_n q^{(\alpha_1\ldots\alpha_n)} m^{-D-2n+2k_1+2k_2}\pi^{-D/2-n}\int d^{D+2n}p\frac{1}{[p^2+m^2]^{k_1}[(p-q)^2+m^2]^{k_2+n}} \\
= \quad & (k_2)_n q^{(\alpha_1\ldots\alpha_n)} G_2\left(k_1, k_2+n, \frac{m^2}{q^2}\right)\Bigg|_{D=4+2n-2\epsilon}. \qquad (67)
\end{aligned}$$

The tensor one-loop integrals one finds in calculating scalar three-loop integrals are *not* given in terms of symmetric and traceless tensors. Instead, one encounters powers of scalar products of various $D$-vectors. However, these latter objects can be reversibly related to contractions of symmetric and traceless tensors. Let us define

$$(p\cdot q)^{(n)} = p^{(\alpha_1\ldots\alpha_n)}q_{(\alpha_1\ldots\alpha_n)}. \qquad (68)$$

In eqns. (A.10) and (A.15) of ref. [11], the following relations have been given:[8]

$$(p\cdot q)^{(n)} = \sum_{\rho=0}^{[n/2]}\frac{(-1)^\rho}{(n+1-\rho-\epsilon)_\rho}\frac{n!}{4^\rho\rho!(n-2\rho)!}(p^2q^2)^\rho(p\cdot q)^{n-2\rho} \qquad (69)$$

$$(p\cdot q)^{n} = \sum_{\rho=0}^{[n/2]}\frac{1}{(n+2-2\rho-\epsilon)_\rho}\frac{n!}{4^\rho\rho!(n-2\rho)!}(p^2q^2)^\rho(p\cdot q)^{(n-2\rho)} \qquad (70)$$

This appendix summarizes all the information on symmetric and traceless tensors necessary for deriving our final expressions (29) and (36) for the three-loop integrals in the most complex case C.2.

[8]There is a misprint in eqn. (A.15) of ref. [11] which we correct here.

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