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$\alpha_s(m_Z)$ from τ decays with matching conditions at three loops

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Abstract

Using the recent four-loop calculations of the QCD β-function and the three-loop matching coefficients we study the induced errors in $\alpha_s(m_Z)$ obtained from $\alpha_s(m_\tau)$ due to the evolution procedure. We show that, when consistent matching and running is used at this order, these errors are pushed below 0.0005 in $\alpha_s(m_Z)$.

The beta function and the quark mass anomalous dimension govern the evolution of the strong coupling constant and the quark masses through the renormalization group (RG) equations,

$$\begin{aligned}
\frac{da}{d\log\mu^2} &= \beta(a) = -a^2\left(\beta_0+\beta_1 a+\beta_2 a^2+\beta_3 a^3\right)+O(a^6)\ , \\
\frac{d\log\bar{m}_q}{d\log\mu^2} &= \gamma_m(a) = -a\left(\gamma_0+\gamma_1 a+\gamma_2 a^2+\gamma_3 a^3\right)+O(a^5)\ ,
\end{aligned} \tag{1}$$

where $a=\alpha_s/\pi$ and $\bar{m}_q$ is the running mass of the quark q. The coefficients of the QCD beta function have been calculated recently in the $\overline{\mathrm{MS}}$ scheme up to four loops [1]

$$\beta_0=\frac{1}{4}\left[11-\frac{2}{3}n_f\right]\ , \qquad \beta_1=\frac{1}{16}\left[102-\frac{38}{3}n_f\right]\ ,$$

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$$\beta_2 = \frac{1}{64}\left[\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right] ,$$
$$\beta_3 = \frac{1}{256}\left[\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right) n_f \right.$$
$$\left. + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) n_f^2 + \frac{1093}{729}n_f^3\right] , \tag{2}$$

and also the coefficients of the quark-mass anomalous dimension have been calculated at the same order [2,3],

$$\gamma_0 = 1 , \qquad \gamma_1 = \frac{1}{16}\left[\frac{202}{3} - \frac{20}{9}n_f\right] ,$$
$$\gamma_2 = \frac{1}{64}\left[1249 + \left(-\frac{2216}{27} - \frac{160}{3}\zeta_3\right) n_f - \frac{140}{81}n_f^2\right] ,$$
$$\gamma_3 = \frac{1}{256}\left[\frac{4603055}{162} + \frac{135680}{27}\zeta_3 - 8800\zeta_5 \right.$$
$$+ \left(-\frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\zeta_4 + \frac{18400}{9}\zeta_5\right) n_f$$
$$\left. + \left(\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right) n_f^2 + \left(-\frac{332}{243} + \frac{64}{27}\zeta_3\right) n_f^3\right] . \tag{3}$$

Here ζ_n is the Riemann zeta-function ($\zeta_2 = \pi^2/6$, $\zeta_3 = 1.202056903\ldots$, $\zeta_4 = \pi^4/90$ and $\zeta_5 = 1.036927755\ldots$) and n_f is the number of quark flavours with mass lower than the renormalization scale μ.

Contrary to what happens in momentum-subtraction schemes (MO) this beta function and the quark mass anomalous dimension are quark mass independent. Thus, the Appelquist-Carazzone theorem [4], that states that eventually the heavy particles decouple at each order in perturbation theory, is not realized in a trivial way since coupling constants, beta functions and quark-mass anomalous dimensions do not exhibit it. To obtain decoupling in the $\overline{\rm MS}$ scheme we need to build in the decoupling region, $\mu \ll m$, with m the mass of the heavy particle, an effective field theory [5] that behaves as if only the light degrees of freedom were present. Matching conditions connect the parameters of the renormalized low-energy effective Lagrangian with the parameters of the full theory. Power suppressed corrections of order $1/m$ contribute to physical observables only through higher order operators but do not affect the matching conditions for the coupling constant and quark masses. The decoupling of the heavy particles is fulfilled in physical quantities once they are expressed in terms of the couplings in the effective theory.

Some time ago it was checked explicitly at three loops [6] that, when the appropriate matching conditions are taken into account, the evolution of the strong coupling constant from low energies to high energies does not depend on the particular choice of the energy scale used to pass a heavy quark threshold. The residual dependences that appear in the perturbative calculation are just an estimate of the effects of the higher order corrections. Very recently [7] the analysis has been extended to four loops and the appropriate coefficients have been computed.

In this paper we use the recently calculated four-loop $\overline{\rm MS}$ scheme QCD beta function and the quark mass anomalous dimension [1–3] to obtain the logarithmic pieces in the matching conditions from a different point of view. Then, we obtain a very convenient analytic form for the running of the QCD coupling constant at four loops and compare it with other solutions. Finally, we use these results and the non-logarithmic coefficients computed in [7] to analyze the impact of the matching conditions on the error induced in $\alpha_s(m_Z)$ if it is obtained from $\alpha_s(m_\tau)$ when passing the thresholds of the c and the b quarks.

Matching conditions in QCD relate the strong coupling constant, a_{n_f}, and the running mass of the light quarks, $\bar{m}_{q,n_f}$, in the full theory with n_f flavours with the effective strong coupling constant, a_{n_f-1}, and the effective light quark masses, $\bar{m}_{q,n_f-1}$, of the effective theory with n_f-1 flavours through a power series in a_{n_f-1}

$$a_{n_f}(\mu_{th}) = a_{n_f-1}(\mu_{th}) \left[1 + \sum_{k=1}^{\infty} C_k(x) a^k_{n_f-1}(\mu_{th})\right] \ , \tag{4}$$

$$\bar{m}_{q,n_f}(\mu_{th}) = \bar{m}_{q,n_f-1}(\mu_{th}) \left[1 + \sum_{k=1}^{\infty} H_k(x) a^k_{n_f-1}(\mu_{th})\right] \ , \tag{5}$$

with coefficients that depend on $x = \log(\mu_{th}^2/m^2)$ where m is some RG-invariant mass of the heavy quark (for instance the RG-invariant-$\overline{\rm MS}$ mass, $\bar{m}(\bar{m})$, or the perturbative pole mass M) that has been integrated out at the energy scale μ_{th}. In order to obtain a good approximation using only the first few terms in the perturbative expansion, we have to evaluate matching conditions in a region where $\mu_{th}/m \sim O(1)$. However, the result of these calculations should not depend on exactly which μ_{th} is chosen.

Note that in contrast to other analysis [7–9] where the effective couplings are expressed in terms of the couplings of the full theory we directly write the inverted relation since we are interested in the evolution of the QCD Lagrangian parameters from low energies to high energies. Note also that in order to simplify as much as possible the matching conditions we have taken as a reference mass, m, a RG invariant mass instead of the running mass $\bar{m}(\mu_{th})$ evaluated at the threshold scale μ_{th}. This makes matching conditions for the α_s's independent of the anomalous dimensions.

The functions C_k and H_k are, in general, polynomials in x. The coefficients multiplying the logarithms of the heavy quark mass are determined just by the RG, that is, they are a function of the beta function and the quark mass anomalous dimension of both the effective theory with n_f-1 flavours and the full theory with n_f flavours. The non-logarithmic coefficients, however, have to be be evaluated explicitly for each particular renormalization scheme.

We apply the renormalization group equations, eq. (1), to both sides of eq. (4) and eq. (5). Identifying order by order in the effective strong coupling constant, a_{n_f-1}, we obtain for the C_k and the H_k functions a set of coupled first-order linear differential equations depending only on the beta and the gamma functions of the full and the effective theories. By solving them, and using the known beta functions we find for the C_k functions

$$
\begin{aligned}
&C_1 = \frac{x}{6}, \qquad C_2 = c_{2,0} + \frac{19}{24}x + \frac{x^2}{36} , \\
&C_3 = c_{3,0} + \left(\frac{241}{54} + \frac{13}{4}c_{2,0} - \left(\frac{325}{1728} + \frac{c_{2,0}}{6}\right) n_f\right) x + \frac{511}{576}x^2 + \frac{x^3}{216} ,
\end{aligned} \tag{6}
$$

while for the H_k functions we obtain

$$
\begin{aligned}
&H_1 = 0, \qquad H_2 = d_{2,0} + \frac{5}{36}x - \frac{x^2}{12} , \\
&H_3 = d_{3,0} + \left(\frac{1627}{1296} - c_{2,0} + \frac{35}{6}d_{2,0} + \left(\frac{35}{648} - \frac{d_{2,0}}{3}\right) n_f + \frac{5}{6}\zeta_3\right) x \\
&\quad - \frac{299}{432}x^2 - \left(\frac{37}{216} - \frac{n_f}{108}\right) x^3 ,
\end{aligned} \tag{7}
$$

where $c_{2,0}$, $c_{3,0}$, $d_{2,0}$ and $d_{3,0}$ are arbitrary constants coming from the integration of the differential equations. They depend on the renormalization scheme and on the RG-invariant reference mass m chosen. They can be determined only by evaluating some Green functions with both the full and the effective theories, in a particular mass-independent renormalization scheme, and then require they are the same, up to terms $O(1/m)$, for values of the renormalization scale just around the threshold. Note that, in order to simplify the results, we have set $c_{1,0} = d_{1,0} = 0$, which is the $\overline{\mathrm{MS}}$ result with the usual dimensional regularization prescription, $\mathrm{Tr}\{\mathrm{I}\} = 4$, with the trace taken in Dirac space.

If the RG-invariant-$\overline{\mathrm{MS}}$ mass is used as a reference scale, that is $m = \bar{m}(\bar{m})$ the coefficients one obtains are [7,9]

$$
c_{2,0} = -\frac{11}{72} , \quad c_{3,0} = \frac{82043}{27648}\zeta_3 - \frac{575263}{124416} + \frac{2633}{31104}n_f , \quad d_{2,0} = -\frac{89}{432} . \tag{8}
$$

If the pole mass is used as a reference scale, that is $m = M$ the coefficients one obtains are [7]

$$
\begin{aligned}
&c_{2,0} = \frac{7}{24} \\
&c_{3,0} = \frac{80507}{27648}\zeta_3 + \frac{1}{9}\zeta_2\left(2\log(2) + 7\right) + \frac{68849}{124416} - \frac{n_f}{9}\left(\zeta_2 + \frac{2479}{3456}\right) .
\end{aligned} \tag{9}
$$

while $d_{2,0}$ does not change, and $d_{3,0}$, when known, has to be shifted by a factor $+10/27$.

Using these coefficients we find complete agreement with ref. [7] also for the logarithmic contributions to the matching for both elections of the reference scale. Note that the authors of ref. [7] present the inverse relations.

For consistency, matching conditions at n loops have to be considered together with running of the $\overline{\mathrm{MS}}$ parameters at $n+1$ loops. The C_3 and the H_3 coefficients depend on at most the three loop beta function and the three loop quark mass anomalous dimension. However, they have to be used together with running at four loops, where the recently calculated four loop beta function and the quark mass anomalous dimension enter the game.

Knowing the beta and gamma functions at four loops we could also obtain the logarithmic pieces of C_4 and H_4, however since the coefficients $c_{4,0}$ and $d_{4,0}$ are not known in any particular scheme and since these matching conditions should be used together with running at five loops, which is also unknown, we do not present them.

In the following we sketch the formulae used to compute the running of α_s at four loops, which is the required order if matching conditions are used at three loops.

We solve the full four-loop RG equation for the strong coupling constant as an expansion in the solution of the two-loop RG equation. In [6] we obtained an expression for the the running of the strong coupling constant as an expansion of the strong coupling constant obtained at one loop. We now improve that expression by resumming some of the leading dependences proportional to β_1. This amounts to expanding around the approximate two-loop solution instead of the one-loop solution.

At the required order we have

$$a(\mu) = a^{(2)}(\mu)\left(1 + c_2(\mu)(a^{(2)}(\mu))^2 + c_3(\mu)(a^{(2)}(\mu))^3\right) \ , \tag{10}$$

where $a^{(2)}(\mu)$ is the approximate two-loop solution

$$a^{(2)}(\mu) = \frac{a(\mu_0)}{K + b_1 a(\mu_0) L + b_1^2 a(\mu_0)^2 (1 - K + L)/K} \tag{11}$$

and

$$\begin{aligned} c_2(\mu) &= b_2(1-K) \ , \\ c_3(\mu) &= \frac{b_3}{2}\left(1-K^2\right) + b_1 b_2 K\left(K-1-L\right) + \frac{b_1^3}{2}\left(L^2 - (1-K)^2\right) \ , \end{aligned} \tag{12}$$

with $K = 1 + \beta_0\, a(\mu_0)\, \log(\mu^2/\mu_0^2)$, $L = \log K$ and $b_k = \beta_k/\beta_0$.

Although these expressions are slightly more complicated than the usual expansion in $1/\log(\Lambda_{QCD})$[7], they are more convenient because the coupling constant at an arbitrary scale is given explicitly in terms of the strong coupling constant at some reference scale μ_0, which usually one takes equal to m_Z or to m_τ. The standard formulae, however, are expressed in terms of

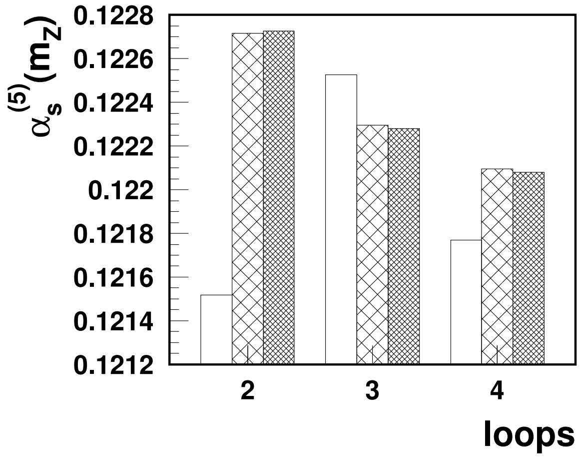

Fig. 1. Values one obtains for $\alpha_s^{(5)}(m_Z)$ starting from $\alpha_s^{(5)}(m_\tau) = 0.336$ at 2,3 and 4 loops by using the different approximations discussed in the text: i) the usual $1/\log(\Lambda_{QCD})$ expansion [7] (white), ii) numerical integration of the renormalization group equation (soft hatching), iii) our expansion in eq. (10) (hard hatching).

Λ_{QCD} which has to be adjusted to the initial conditions for α_s. So, the value of Λ_{QCD} to be used depends on the number of light flavours considered and on the number of loops considered. Moreover it seems that the $1/\log(\Lambda_{QCD})$ expansion converges more slowly than our expansion. We show in fig. 1 the values obtained for $\alpha_s^{(5)}(m_Z)$ starting from $\alpha_s^{(5)}(m_\tau) = 0.336$ by using three different solutions of the RG at 2,3 and 4 loops : i) the usual $1/\log(\Lambda_{QCD})$ expansion [7] (white), ii) numerical integration of the renormalization group equation (soft hatching), iii) our expansion in eq. (10) (hard hatching). Clearly our expansion gives much closer results to the numerical integration and seems to converge faster than the $1/\log(\Lambda_{QCD})$ expansion. Therefore we will use it in the following.

Let us turn to the phenomenological applications of the matching conditions and the four-loop running solutions.

The most precise determinations of α_s are obtained from hadronic τ decays [10] at rather low energies and from hadronic Z decays at LEP energies [11] . To compare these two results one has to connect the strong coupling constant in a theory with three flavours at a scale $\mu = m_\tau$ with the strong coupling constant in a theory with five flavours at a scale $\mu = m_Z$. Therefore, two thresholds have to be passed, the threshold of the c-quark and the threshold of the b-quark. Moreover, running between a wide range of scales has to be performed. Note that, although $m_c < m_\tau$, results for $\alpha_s(m_\tau)$ are usually presented in a theory with only three quark flavours. That is because c-quarks cannot be really produced in τ decays and they only enter in loops. Therefore it is appropriate to use an effective theory in which the c-quark has been integrated out. Power corrections of the form m_τ^2/m_c^2 can be included in the effective theory and have been computed [12,9]. They are very small and they are taken into account in the extracted value of $\alpha_s(m_\tau)$.

Given the present accuracy of both experimental measurements and theoretical calculations of hadronic decays of the τ lepton, it is important to calculate very precisely the connection between coupling constants when passing thresholds and to estimate the remaining errors in the calculation. It is in this analysis where the three loop-matching conditions for α_s and four-loop running studied above are relevant.

In the following we obtain $\alpha_s(m_Z)$ by using as starting point $\alpha_s(m_\tau)$ and we estimate the residual errors due to the matching conditions and running for the different approximations used. We follow the same procedure as in [6]. At low energies, $\mu = m_\tau = 1777.0 \pm 0.3\ MeV$ we know [10] $\alpha_s(m_\tau) \equiv \alpha_s^{(3)}(m_\tau) = 0.35 \pm 0.02$. From this we can obtain $\alpha_s^{(3)}(\mu_{th}^c)$ at some matching point μ_{th}^c around $\bar{m}_c$ by using the renormalization group with $n_f = 3$, then, by using eq. (4) with $m = \bar{m}_c$ and $n_f = 4$ we obtain $\alpha_s^{(4)}(\mu_{th}^c)$. Now we use again the renormalization group with $n_f = 4$ to obtain $\alpha_s^{(4)}(\mu_{th}^b)$ at some matching point μ_{th}^b around $\bar{m}_b$ and use again eq. (4) with $m = \bar{m}_b$ and $n_f = 5$ to obtain $\alpha_s^{(5)}(\mu_{th}^b)$. Finally we use the renormalization group with $n_f = 5$ to obtain $\alpha_s^{(5)}(m_Z) \equiv \alpha_s(m_Z)$. The final result will depend on the precise values used for μ_{th}^c and μ_{th}^b and this dependence gives an estimate of the errors which arise because the truncation of the perturbative series in the matching conditions. In addition, matching conditions also depend on the masses of the quarks, and, although they are very well known, their actual value can affect the final result for $\alpha_s(m_Z)$. The induced error due to the uncertainty in the quark masses is dominated by the one-loop matching equation. Then, we can estimate this error as $(\Delta\alpha_s(m_Z))/\alpha_s(m_Z) \approx \alpha_s(m_q)/(3\pi)(\Delta m_q)/m_q$. We use always as a reference scale the RG-invariant-$\overline{\mathrm{MS}}$ mass of the quarks, and therefore the coefficients in eq. (8). For the quark masses we take the last values in the literature: for the b-quark mass $\bar{m}_b(\bar{m}_b) = 4.13 \pm 0.06\ GeV$ [13]. For the c-quark mass we take $\bar{m}_c(\bar{m}_c) = 1.31 \pm 0.06\ GeV$, [14] (see also [15]).

We study the effect of varying the scale at which matching is performed independently for the c and the b quarks.

First we fix $\mu_{th}^c = \bar{m}_c$ and vary μ_{th}^b in $2 - 20$ GeV. Figure 2 shows (dotted line) our result for two-loop running (only the first term of eq. (10) is taken into

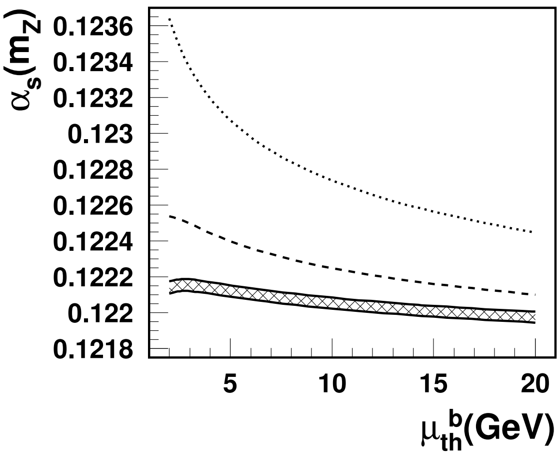

Fig. 2. $\alpha_s(m_Z)$ obtained by running the coupling from $\alpha_s(m_\tau) = 0.35$, as a function of the matching point taken to cross the b-quark threshold. Dotted line: two-loop beta functions and one-loop matching conditions. Dashed line: three-loop beta functions and two-loop matching conditions. The hatched band is obtained with four-loop beta functions and three-loop matching conditions when the b-quark mass is varied within its error interval.

account) and one-loop matching conditions (only the C_1 coefficient is considered). We plot with dashed line the results for three-loop running and two-loop matching conditions (two terms in eq. (10) and C_2 included in eq. (4)). For these two lines we took central values for the b-quark mass. Finally, the hatched area gives the results for four-loop running and three-loop matching conditions when the b-quark mass is varied within its error interval. For the central value of the strong coupling constant extracted from tau decays we find that varying the b-quark threshold scale in the range $\mu_{th}^b = 2 - 20\ GeV$ two-loop running and one-loop matching conditions induce an error of 0.0006 on the strong coupling constant at the Z-boson mass scale. With three-loop running and two-loop matching conditions the error decreases to 0.0002. For four-loop running and three-loop matching conditions we get and error for the central value of the b-quark mass of 0.00009. The uncertainty in the b-quark mass induces in this case an additional error of 0.00003.

To study the errors induced in passing the c-quark threshold we fix $\mu_{th}^b = \bar{m}_b$ and vary μ_{th}^c in the range $\mu_{th}^c = 1 - 4 GeV$. Then, we find an induced error of 0.0005, 0.0002 and 0.0001 to each order respectively. The uncertainty in the c-quark mass introduces an additional error of 0.0001.

To analyze the combined effect of passing the two thresholds we have represented the value of $\alpha_s(m_Z)$ obtained at four loops as a function of μ_{th}^c and μ_{th}^b as a contour plot in fig. 3 by taking $\log(\mu_{th}^b/1GeV)$ and $\log(\mu_{th}^c/1GeV)$ in the x and y axis respectively. The different contour lines are obtained for $\alpha_s(m_Z) = 0.12214$ to $\alpha_s(m_Z) = 0.12182$ in steps of 0.00004. We see that there

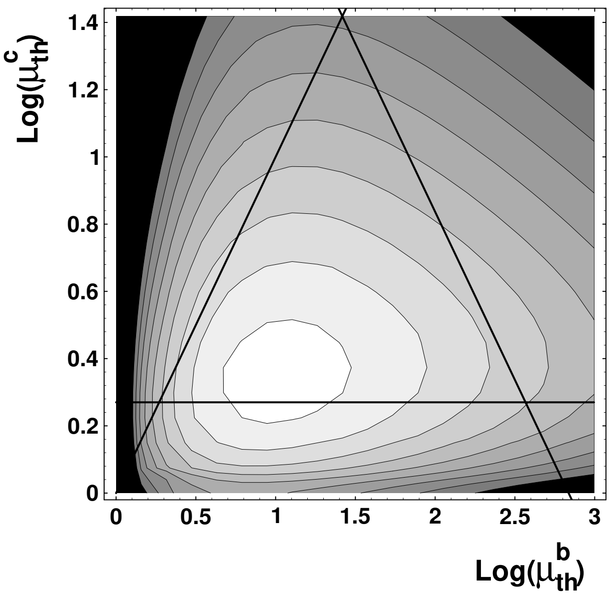

Fig. 3. $\alpha_s(m_Z)$ as a function of the matching point scales μ^c_{th} and μ^b_{th}. The contours are for $\alpha_s(m_Z) = 0.12214$ to $\alpha_s(m_Z) = 0.12182$ in steps of 0.00004. The area within the strait lines is obtained by taking $\bar{m}_c < \mu^c_{th} < \mu^b_{th} < \bar{m}_b^2/\mu^c_{th}$.

is a maximum approximately for $\alpha_s(m_Z) = 0.1222$. The minimum depends on how far away from $\bar{m}_b$ and $\bar{m}_c$ we take the matching scales μ^b_{th} and μ^c_{th}. We choose $\bar{m}_c < \mu^c_{th} < \mu^b_{th} < \bar{m}_b^2/\mu^c_{th}$, which are represented by the three strait lines in fig. 3. Then we obtain an estimate of the error due to the unknowledge of the scales μ^c_{th} and μ^b_{th} of about $\Delta\alpha_s(m_Z) = 0.0002$ by taking half the difference between the maximum and the minimum values of $\alpha_s(m_Z)$. If in addition to this we also consider the error induced because the uncertainty in the input values of the masses of the c and the b quarks, we obtain that the total error in $\alpha_s(m_Z)$ due to the matching procedure is $\Delta\alpha_s(m_Z) = 0.0003$. Finally we can also include the error in the extracted value of $\alpha_s(m_\tau)$, which is the dominant one. We obtain the values of $\alpha_s(m_Z)$ starting from $\alpha_s(m_\tau) = 0.35 \pm 0.02$ and take the average of the maximum and minimum to get the central value and half of the difference to obtain the error. This gives $\Delta\alpha_s(m_Z) = 0.0021$. Combining lineally all errors we obtain

$$\alpha_s(m_Z) = 0.1219 \pm 0.0024 \tag{13}$$

Clearly, considering matching conditions at three loops and running at four loops reduces the errors due to the matching procedure by a factor two and renders them below 0.0005 in $\alpha_s(m_Z)$. This has to be compared with the full error which is of about 0.0025.

By using the recent four-loop QCD calculations we have obtained the matching conditions for $\alpha_s(\mu)$ and $\bar{m}_q(\mu)$ when crossing a quark threshold. Then, we have solved the renormalization group equations at four-loops and have obtained a very convenient analytic form for the running of the QCD coupling constant.

We have compared the results obtained by using this solution with the results obtained by using the usual $1/\log(\Lambda_{QCD})$ expansion and with the numerical solution at the different orders and found that it gives a good approximation to the numerical solution and converges faster than the usual $1/\log(\Lambda_{QCD})$ expansion. Finally we have used these results to study the effect of the c- and the b-quark thresholds in the evolution of the strong coupling constant from $\mu = m_\tau$ to $\mu = m_Z$. This analysis is very important given the present accuracy of both determinations of the QCD coupling constant, $\alpha_s(m_\tau)$ and $\alpha_s(m_Z)$. We found that the total error induced in $\alpha_s(m_Z)$ starting from $\alpha_s(m_\tau)$ due to the matching and running procedures is 0.0003 when matching and running evolution are used at four loops.

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