Quadratic Mass Corrections of Order $\mathcal{O}(\alpha_s^3 m_q^2/s)$ to the Decay Rate of Z- and W- Bosons

K.G. Chetyrkin, [a,b] J. H. Kühn [c]

[a] *Institute for Nuclear Research, Russian Academy of Sciences, 60th October Anniversary Prospect 7a, Moscow 117312, Russia*

[b] *Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany*

[c] *Institut für Theoretische Teilchenphysik, Universität Karlsruhe D-76128 Karlsruhe, Germany*

Abstract

We analytically compute quadratic mass corrections of order $\mathcal{O}(\alpha_s^3 m_q^2/s)$ to the absorptive part of the (non-diagonal) correlator of two axial vector currents. This allows us to find the correction of order $\mathcal{O}(\alpha_s^3 m_q^2/M_W^2)$ to $\Gamma(W \to \text{hadrons})$ as well as similar corrections to $\Gamma(Z \to \text{hadrons})$.

PACS numbers: 12.38.-t; 12.38.Bx; 13.38.Be; 13.38.Dg

Keywords: QCD; Z boson decay; W boson decay;

Corresponding author: K.G. Chetyrkin, address: Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany
e-mail: chet@mppmu.mpg.de

1 Introduction

Precision measurements of the total and as well as partial Z decay rates have provided one of the the most important and, from the theoretical viewpoint, clean determination of the strong coupling constant α_s with a present value of $\alpha_s = 0.1202 \pm 0.0033$ [1]. Theoretical ingredients were the knowledge of QCD corrections to order α_s^3 in the limit of massless quarks plus charm and bottom quarks effects (see, e.g. [2] and references therein). These mass corrections which indeed are relevant at the present level of accuracy have been calculated up to the order $\alpha_s^3 m_q^2/s$ for the vector and $\alpha_s^2 m_q^2/s$ for the axial current induced decay rate. In this short note the prediction is extended to include $\alpha_s^3 m_q^2/s$ terms for the (non-singlet part) of the axial current induced rate. At the same time results are obtained for the non-diagonal current correlator with two different masses – a case of relevance e.g. for the W decay rate into charmed and bottom quarks. The same formulae can also be applied to a subclass of corrections which enter single top production in the Drell-Yan like reaction $q\bar{q} \to t\bar{b}$ far above threshold.

The calculation is based on an approach introduced in Refs. [3,4]. Knowledge of the polarization function to order α_s^2, the appropriate anomalous dimensions at order α_s^3, combined with the renormalization group equation allows one to predict the corresponding logarithmic terms of order α_s^3 and hence the constant terms of the imaginary part. The first of these ingredients has been available since some time [5–8] while the anomalous dimension can been obtained from Ref. [9] in a straightforward way.

In this short note only the theoretical framework and the analytical results are presented – numerical studies will presented elsewhere.

2 Renormalization Group Analysis

In analogy to the vector case, we take as a starting point the generic vector/axial quark current correlator $\Pi_{\mu\nu}^{V/A}$ which is defined by

$$\begin{aligned}\Pi_{\mu\nu}^{V/A}(q, m_u, m_d, m, \mu, \alpha_s) &= i \int dx e^{iqx} \langle T[\, j_\mu^{V/A}(x) (j_\nu^{V/A})^\dagger(0)\,]\rangle \\ &= g_{\mu\nu}\Pi_{V/A}^{(1)}(Q^2) + q_\mu q_\nu \Pi_{V/A}^{(2)}(Q^2)).\end{aligned} \tag{1}$$

with $Q^2 = -q^2$, $m^2 = \sum_f m_f^2$ and $j_\mu^{V/A} = \bar{q}\gamma_\mu(\gamma_5)q'$. Here q and q' are two (generically different) quarks with masses m_u and m_d respectively. Note that the vector and axial correlators are related through

$$\Pi_{\mu\nu}^{A}(q, m_u, m_d, m, \mu, \alpha_s) = \Pi_{\mu\nu}^{V}(q, m_u, -m_d, m, \mu, \alpha_s) \tag{2}$$

The polarization function $\Pi^{(1)}_{V/A}$ and the spectral density $R^{V/A}(s)$ which in turn governs the Z and W decays rate obey the following dispersion relation

$$\Pi^{(1)}_{V/A}(Q^2) = \frac{-1}{12\pi^2} \int\limits_{(m_u+m_d)^2}^{\infty} ds \frac{sR^{V/A}(s, m_u, m_d, m, \mu, \alpha_s)}{s + Q^2} \qquad \text{mod sub.} \tag{3}$$

Whereas $R^{V/A}$ as a physical quantity is invariant under renormalization group transformations, the function $\langle T[\, j^{V/A}_\mu(x)(j^{V/A}_\nu)^\dagger(0)\,]\rangle$ contains some non-integrable singularities in the vicinity of the point $x = 0$. These cannot be removed by standard quark mass and coupling constant renormalizations, but must be subtracted independently. As a result the relevant renormalization group equation assumes the form [2]

$$\mu^2 \frac{d}{d\mu^2} \Pi^{V/A}_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu} q^2)\gamma^{\pm}_q(\alpha_s)\frac{1}{16\pi^2} + (m_u \mp m_d)^2 g_{\mu\nu}\gamma^{\pm}_m(\alpha_s)\frac{1}{16\pi^2}, \tag{4}$$

where

$$\mu^2 \frac{d}{d\mu^2} = \mu^2 \frac{\partial}{\partial\mu^2} + \pi\beta(\alpha_s)\frac{\partial}{\partial\alpha_s} + \gamma_m(\alpha_s)\sum_f \bar{m}_f \frac{\partial}{\partial\bar{m}_f}. \tag{5}$$

Here and below the upper and lower signs give the results for vector and axial vector correlators respectively. From the identity (2) we infer that both anomalous dimensions $\gamma^{\pm}_q$ and $\gamma^{\pm}_m$, being not dependent on any masses, also do not depend on the sign. In what follows we will denote

$$\gamma^{\pm}_q = \gamma^{VV}_q \quad \text{and} \quad \gamma^{\pm}_m = \gamma^{VV}_m \ .$$

The β-function and the quark mass anomalous dimension γ_m are defined in the usual way

$$\mu^2 \frac{d}{d\mu^2}\left(\frac{\alpha_s(\mu)}{\pi}\right) = \beta(\alpha_s) \equiv -\sum_{i\geq 0} \beta_i \left(\frac{\alpha_s}{\pi}\right)^{i+2}, \tag{6}$$

$$\mu^2 \frac{d}{d\mu^2}\bar{m}(\mu) = \bar{m}(\mu)\gamma_m(\alpha_s) \equiv -\bar{m}\sum_{i\geq 0} \gamma^i_m \left(\frac{\alpha_s}{\pi}\right)^{i+1}. \tag{7}$$

Their expansion coefficients up to order $\mathcal{O}(\alpha_s^3)$ are well known [10–13] and read (n_f is the number of quark flavours)

$$\begin{aligned} \beta_0 &= \left(11 - \frac{2}{3}n_f\right)/4, \quad \beta_1 = \left(102 - \frac{38}{3}f\right)/16, \\ \beta_2 &= \left(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right)/64, \end{aligned} \tag{8}$$

$$
\begin{aligned}
\gamma_m^0 = 1, \quad \gamma_m^1 &= \left(\frac{202}{3} - \frac{20}{9} f\right)/16, \\
\gamma_m^2 = \left(1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] n_f - \frac{140}{81} f^2\right)/64.
\end{aligned}
\tag{9}
$$

Another useful and closely related object is the correlator of the (pseudo)scalar quark currents

$$
\Pi^{\mathrm{S/P}}(Q^2, m_u, m_d, m, \mu, \alpha_s) = \int e^{iqx} \langle 0|\; T\; [\, j^{\mathrm{S/P}}(x) (j^{\mathrm{S/P}})^\dagger(0)\,]\; |0\rangle \,. \tag{10}
$$

Scalar and pseudoscalar current correlators are also related in a simple manner:

$$
\Pi^{\mathrm{S}}(Q^2, m_u, m_d, m, \mu) = \Pi^{\mathrm{P}}(Q^2, m_u, -m_d, m, \mu). \tag{11}
$$

For vanishing quark masses scalar and pseudoscalar correlators are, therefore, identical: $\Pi^{\mathrm{S}} = \Pi^{\mathrm{P}}$ and meet the following RG equation

$$
\left(\mu^2 \frac{d}{d\mu^2} + 2\gamma_m(\alpha_s)\right) \Pi^{\mathrm{S/P}} = Q^2 \gamma_q^{\mathrm{SS}}(\alpha_s) \frac{1}{16\pi^2}. \tag{12}
$$

The (axial) vector and (pseudo)scalar correlators are connected through a Ward identity [14]

$$
q_\mu q_\nu \Pi_{\mu\nu}^{\mathrm{V/A}} = (m_u \mp m_d)^2 \Pi^{\mathrm{S/P}} + (m_u \mp m_d)(\langle \overline{\psi}_{\mathrm{q}} \psi_{\mathrm{q}} \rangle \mp \langle \overline{\psi}_{\mathrm{q'}} \psi_{\mathrm{q'}} \rangle)\,, \tag{13}
$$

where the vacuum expectation values on the r.h.s. are understood within the framework of perturbation theory and the minimal subtractions. Equation (13) leads to the following relation between the corresponding anomalous dimensions [4]:

$$
\gamma_m^{\mathrm{VV}} \equiv -\gamma_q^{\mathrm{SS}}\,. \tag{14}
$$

This relation was used in Ref. [4] in order to find the anomalous dimension γ_m^{AA} at the α_s^2 order starting from the results of Ref. [5].

In what follows we will be interested in quadratic mass corrections to the polarization operator $\Pi_A^{(1)}$ which is convenient to represent in the form ($\mathbf{m} = \{m_u, m_d, m\}$):

$$
\Pi_{V/A}^{(1)}(Q^2, \mathbf{m}, \mu, \alpha_s) = \frac{3}{16\pi^2} \Pi_{V/A,0}^{(1)}(Q^2, \mu, \alpha_s) + \frac{3}{16\pi^2} \Pi_{V/A,2}^{(1)}(\frac{\mu^2}{Q^2}, \mathbf{m}, \alpha_s) + \mathcal{O}(\mathbf{m}^4). \tag{15}
$$

Here the first term on the rhs corresponds to the massless limit while the second term stands for quadratic mass corrections. Note that $\Pi_{V/A,2}^{(1)}$ is a second order polynomial in quark masses: a logarithmic dependence on quark masses may appear starting from m^4 terms only [1]. Working only within the second order in quark masses, the vacuum expectation values on the right hand side of (13) can be safely discarded. As a result, the latter can be written as follows:

$$-\Pi^{(1)}_{V/A,2} + Q^2\Pi^{(2)}_{V/A,2} = \frac{(m_u \mp m_d)^2}{Q^2}\Pi^{\mathrm{S/P},0}, \tag{16}$$

with $\Pi^{\mathrm{S/P},0}$ being the massless scalar correlator.

From the RG equation (4) we arrive at the following equation for $\Pi^{(1)}_{V/A,2}$:

$$\mu^2\frac{d}{d\mu^2}\Pi^{(1)}_{V/A,2} = \frac{1}{3}(m_u \mp m_d)^2\gamma_m^{VV}(\alpha_s) \tag{17}$$

or, equivalently, ($L_q = \ln\frac{\mu^2}{Q^2}$)

$$\frac{\partial}{\partial L_q}\Pi^{(1)}_{V/A,2} = \frac{1}{3}(m_u \mp m_d)^2\gamma_m^{VV} - \left(\beta\frac{\partial}{\partial\alpha_s} + 2\gamma_m\right)\Pi^{(1)}_{V/A,2}. \tag{18}$$

The last relation explicitly demonstrates that $R_2^{V/A}$ — the absorptive part of $\Pi^{(1)}_{V/A,2}$ — depends in order α_s^n on the very function $\Pi^{(1)}_{V/A,2}$ which is multiplied by at least one factor of α_s. This means that one needs to know $\Pi^{(1)}_{V/A,2}$ up to order α_s^{n-1} only to unambiguously reconstruct all Q-dependent terms in $\Pi^{(1)}_{V/A,2}$ to α_s^n, provided, of course, the anomalous dimensions γ_m and γ_m^{VV} are known to α_s^n and the beta function to α_s^{n+1}.

This observation was made first in [3] where it was used to find the absorptive part R_2^V in order α_s^3 for the case of the diagonal vector current (that is for the case of $m_u = m_d$). In the present paper we will use the results of a recent calculation of γ_q^{SS} [9] to order α_s^3 to determine the absorptive part $R_2^{V/A}$ to the same order in the general case of non-diagonal currents.

3 Calculation and results

The result for the function $\Pi^{(2)}_{V/A,2}$ in the general non-diagonal case to order α_s^2 was first published in Ref. [6]. On the other hand, the identity (16) expresses the combination $-\Pi^{(1)}_{V/A,2}/Q^2 + \Pi^{(2)}_{V/A,2}$ in terms of the *massless* polarization operator Π^S known from Refs. [5,7]. A sum of these two functions leads us to the following result for $\Pi^{(1)}_{V,2}$

[1] Provided of course that one uses a mass independent renormalization scheme like the $\overline{\mathrm{MS}}$-scheme employed in this work.

$$
\begin{aligned}
\Pi^{(1)}_{V,2} &= m_-^2\left[2+2\ln\frac{\mu^2}{Q^2}\right] + m_+^2\,[-2] \\
&+ m_-^2\frac{\alpha_s}{\pi}\left[\frac{107}{6} - 8\,\zeta(3) + \frac{22}{3}\ln\frac{\mu^2}{Q^2} + 2\ln^2\frac{\mu^2}{Q^2}\right] + m_+^2\frac{\alpha_s}{\pi}\left[-\frac{16}{3} - 4\ln\frac{\mu^2}{Q^2}\right] \\
&+ m_-^2\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{3241}{18} - 129\,\zeta(3) - \frac{1}{2}\,\zeta(4) + 55\,\zeta(5) - \frac{857}{108}\,n_f + \frac{32}{9}\,\zeta(3)\,n_f\right. \\
&\qquad + \frac{8221}{72}\ln\frac{\mu^2}{Q^2} - 39\,\zeta(3)\ln\frac{\mu^2}{Q^2} - \frac{151}{36}\,n_f\ln\frac{\mu^2}{Q^2} + \frac{4}{3}\,\zeta(3)\,n_f\ln\frac{\mu^2}{Q^2} \\
&\qquad \left. + \frac{155}{6}\ln^2\frac{\mu^2}{Q^2} - \frac{8}{9}\,n_f\ln^2\frac{\mu^2}{Q^2} + \frac{19}{6}\ln^3\frac{\mu^2}{Q^2} - \frac{1}{9}\,n_f\ln^3\frac{\mu^2}{Q^2}\right] \\
&+ m_+^2\left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{19691}{216} - \frac{124}{27}\,\zeta(3) + \frac{1045}{27}\,\zeta(5) + \frac{95}{36}\,n_f - \frac{253}{6}\ln\frac{\mu^2}{Q^2}\right. \\
&\qquad \left. + \frac{13}{9}\,n_f\ln\frac{\mu^2}{Q^2} - \frac{19}{2}\ln^2\frac{\mu^2}{Q^2} + \frac{1}{3}\,n_f\ln^2\frac{\mu^2}{Q^2}\right] \\
&+ m^2\left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{128}{9} - \frac{32}{3}\,\zeta(3)\right]. \qquad (19)
\end{aligned}
$$

Here $m_- = m_u - m_d$ and $m_+ = m_u + m_d$, $Q^2 = -q^2$, all masses as well as QCD coupling constant α_s are understood to be taken at a generic value of the t' Hooft mass μ. All correlators are renormalized within $\overline{\mathrm{MS}}$-scheme. We have also checked (19) by a direct calculation with the help of the program MINCER [16] written for the symbolic manipulation system FORM [17]. In a particular case of $m_u = m_d$ Eq. (19) is in agreement with Refs. [8,15].

Now, as was shown in [3] the anomalous dimension $\gamma_m^{AA} \equiv= -\gamma_q^{SS}$, and, thus, from the results of [9] we have:

$$
\begin{aligned}
\gamma_m^{VV} &= -\gamma_q^{SS} = 6\left\{1+\frac{5}{3}\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{455}{72} - \frac{1}{2}\,\zeta(3) - \frac{1}{3}\,n_f\right]\right. \\
&+ \left(\frac{\alpha_s}{\pi}\right)^3\left[\frac{157697}{5184} - \frac{1645}{216}\,\zeta(3) + \frac{15}{8}\,\zeta(4) + \frac{65}{12}\,\zeta(5) - \frac{14131}{7776}\,n_f\right. \\
&\qquad \left.\left. - \frac{13}{9}\,\zeta(3)\,n_f - \frac{11}{12}\,\zeta(4)\,n_f - \frac{1625}{11664}\,n_f^2 + \frac{1}{9}\,\zeta(3)\,n_f^2\right]\right\}. \qquad (20)
\end{aligned}
$$

At last, integrating eq. (18) we find the spectral density R_2^V in general case to order α_s^3:

$$
R_2^V(s, m_u, m_d, m, \mu, \alpha_s) = 3\left\{\frac{m_+^2}{s}r^V_{2,+} + \frac{m_-^2}{s}r^V_{2,-} + \frac{m^2}{s}r^V_{2,0}\right\}, \qquad (21)
$$

where the functions r^V are

$$
\begin{aligned}
r^V_{2,+} = {}& 3\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{253}{8} - \frac{13}{12}\,n_f + \frac{57}{4}\ln\frac{\mu^2}{s} - \frac{1}{2}\,n_f\ln\frac{\mu^2}{s}\right] \\
&+ \left(\frac{\alpha_s}{\pi}\right)^3\left[\frac{1261}{2} - \frac{285}{16}\pi^2 + \frac{155}{6}\,\zeta(3) - \frac{5225}{24}\,\zeta(5) - \frac{2471}{54}\,n_f + \frac{17}{12}\pi^2\,n_f\right. \\
&\qquad - \frac{197}{54}\,\zeta(3)\,n_f + \frac{1045}{108}\,\zeta(5)\,n_f + \frac{125}{216}\,n_f^2 - \frac{1}{36}\pi^2\,n_f^2 + \frac{4505}{16}\ln\frac{\mu^2}{s} \\
&\qquad \left. - \frac{175}{8}\,n_f\ln\frac{\mu^2}{s} + \frac{13}{36}\,n_f^2\ln\frac{\mu^2}{s} + \frac{855}{16}\ln^2\frac{\mu^2}{s} - \frac{17}{4}\,n_f\ln^2\frac{\mu^2}{s} + \frac{1}{12}\,n_f^2\ln^2\frac{\mu^2}{s}\right],
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
r^V_{2,-} = {}& -\frac{3}{2} + \frac{\alpha_s}{\pi}\left[-\frac{11}{2} - 3\ln\frac{\mu^2}{s}\right] \\
&+ \left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{8221}{96} + \frac{19}{8}\pi^2 + \frac{117}{4}\,\zeta(3) + \frac{151}{48}\,n_f - \frac{1}{12}\pi^2\,n_f\right. \\
&\qquad \left. - \zeta(3)\,n_f - \frac{155}{4}\ln\frac{\mu^2}{s} + \frac{4}{3}\,n_f\ln\frac{\mu^2}{s} - \frac{57}{8}\ln^2\frac{\mu^2}{s} + \frac{1}{4}\,n_f\ln^2\frac{\mu^2}{s}\right] \\
&+ \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{4544045}{3456} + \frac{335}{6}\pi^2 + \frac{118915}{144}\,\zeta(3) - \frac{635}{2}\,\zeta(5) + \frac{71621}{648}\,n_f\right. \\
&\qquad - \frac{209}{48}\pi^2\,n_f - 54\,\zeta(3)\,n_f + \frac{5}{4}\,\zeta(4)\,n_f + \frac{55}{4}\,\zeta(5)\,n_f - \frac{13171}{7776}\,n_f^2 \\
&\qquad + \frac{2}{27}\pi^2\,n_f^2 + \frac{13}{18}\,\zeta(3)\,n_f^2 - \frac{4693}{6}\ln\frac{\mu^2}{s} + \frac{285}{16}\pi^2\ln\frac{\mu^2}{s} + \frac{1755}{8}\,\zeta(3)\ln\frac{\mu^2}{s} \\
&\qquad + \frac{8909}{144}\,n_f\ln\frac{\mu^2}{s} - \frac{17}{12}\pi^2\,n_f\ln\frac{\mu^2}{s} - \frac{59}{4}\,\zeta(3)\,n_f\ln\frac{\mu^2}{s} - \frac{209}{216}\,n_f^2\ln\frac{\mu^2}{s} \\
&\qquad + \frac{1}{36}\pi^2\,n_f^2\ln\frac{\mu^2}{s} + \frac{1}{3}\,\zeta(3)\,n_f^2\ln\frac{\mu^2}{s} - \frac{335}{2}\ln^2\frac{\mu^2}{s} + \frac{209}{16}\,n_f\ln^2\frac{\mu^2}{s} \\
&\qquad \left. - \frac{2}{9}\,n_f^2\ln^2\frac{\mu^2}{s} - \frac{285}{16}\ln^3\frac{\mu^2}{s} + \frac{17}{12}\,n_f\ln^3\frac{\mu^2}{s} - \frac{1}{36}\,n_f^2\ln^3\frac{\mu^2}{s}\right],
\end{aligned}
\tag{23}
$$

$$
r^V_{2,0} = \left(\frac{\alpha_s}{\pi}\right)^3\left[-80 + 60\,\zeta(3) + \frac{32}{9}\,n_f - \frac{8}{3}\,\zeta(3)\,n_f\right].
\tag{24}
$$

The expressions for the hadronic decay rates of the intermediate bosons read:

$$
\begin{aligned}
\Gamma(Z \to \text{hadrons}) = \Gamma_0^Z\Big[&\sum_f((g_V^f)^2 + (g_A^f)^2)\left(R_0(s) + R_2^V(s,0,0,\sqrt{m_b^2+m_c^2}\right) \\
&+ \sum_{f=b,c}(g_V^f)^2 R_2^V(s,m_f,m_f,0) \\
&+ \sum_{f=b,c}(g_A^f)^2 R_2^A(s,m_f,m_f,0)\Big],
\end{aligned}
\tag{25}
$$

$$\Gamma(W \to \text{hadrons}) = \Gamma_0^W \Big\{ 2\left[R_0(s) + R_2^V\left(s,0,0,\sqrt{m_b^2+m_c^2}\right)\right] + \frac{1}{2} \sum_{\substack{i=u,c\\ j=d,s,b}} |V_{i,j}|^2 (R_2^V(s,m_i,m_j,0) + R_2^A(s,m_i,m_j,0)) \Big\}, \tag{26}$$

with $\Gamma_0 = \frac{G_F M_{Z/W}^3}{6\pi\sqrt{2}}$, $g_V^f = I_3^f - 2Q_f \sin^2\theta_{\rm w}$, $g_A^f = I_3^f$ and V_{ij} being the CKM matrix.

Here $R_0(s)$ is the (non-singlet part) of the ratio $R(s)$ in massless QCD; it was computed to α_s^3 in [18,19] and confirmed in [20]; it reads:

$$\begin{aligned} R_0(s) = 3\Big\{ 1 &+ \frac{\alpha_s}{\pi} \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{365}{24} - 11\zeta(3) + n_f\left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right) + \left(-\frac{11}{4} + \frac{1}{6}n_f\right)\ln\frac{s}{\mu^2}\right] \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3 \Big[\frac{87029}{288} - \frac{121}{48}\pi^2 - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) \\ &\quad + n_f\left(-\frac{7847}{216} + \frac{11}{36}\pi^2 + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right) + n_f^2\left(\frac{151}{162} - \frac{1}{108}\pi^2 - \frac{19}{27}\zeta(3)\right) \\ &\quad + \left(-\frac{4321}{48} + \frac{121}{2}\zeta(3) + n_f\left[\frac{785}{72} - \frac{22}{3}\zeta(3)\right] + n_f^2\left[-\frac{11}{36} + \frac{2}{9}\zeta(3)\right]\right)\ln\frac{s}{\mu^2} \\ &\quad + \left(\frac{121}{16} - \frac{11}{12}n_f + \frac{1}{36}n_f^2\right)\ln^2\frac{s}{\mu^2}\Big]\Big\}. \end{aligned}$$

In deriving Eqs. (25) and (26) we have assumed that (i) the top quark is completely decoupled (the power suppressed corrections to this approximation start from the order $\frac{s}{m_t^2}\alpha_s^2$ and have been studied in Refs. [22–25,25]); (ii) all other quarks except for the charmed and bottom ones are massless. Note that for the case of *diagonal* currents there exist also so-called singlet contributions to $R(s)$. We will ignore these contributions in what follows as they are absent for the case of non-diagonal currents relevant for the W-decay (a detailed discussion of the Z-decay rate including singlet contributions can be found in [2]).

Taking into account the peculiar structure of the general result (21), the last formula can be written in a simpler form, viz.

$$\begin{aligned} \Gamma(W \to \text{hadrons}) = \Gamma_0^W \Big[\; & 2\left(R_0(s) + R_2^V(s,0,0,\sqrt{m_b^2+m_c^2})\right) \\ & + R_2^V(s, m_{eff}, 0, 0) \Big]. \end{aligned} \tag{27}$$

Here

$$m_{eff}^2 = |V_{c,b}|^2(m_c^2 + m_b^2) + |V_{u,b}|^2 m_b^2$$

and we have taken into account the fact that

$$R^V(s, m_i, m_j, 0) + R^A(s, m_i, m_j, 0) = 2R^V(s, \sqrt{m_i^2 + m_j^2}, 0, 0)$$
$$= 2R^A(s, \sqrt{m_i^2 + m_j^2}, 0, 0).$$

As a direct consequence of Eqs. (2,21) we obtain the following expressions for particular functions entering into (25,26)

$$R_2^V(s, m, m, 0) = \frac{4m^2}{s} 3\, r^V_{2,+}, \tag{28}$$

$$R_2^A(s, m, m, 0) = \frac{4m^2}{s} 3\, r^V_{2,-}, \tag{29}$$

$$R_2^V(s, m, 0, 0) = R_2^A(s, m, 0, 0) = \frac{m^2}{s} 3\, (r^V_{2,+} + r^V_{2,-}), \tag{30}$$

$$R_2^V(s, 0, 0, m) = R_2^A(s, 0, 0, m) = \frac{m^2}{s} 3\, r_{2,0}. \tag{31}$$

At last, with $n_f = 5$ and $\mu^2 = s$ the above formulas are simplified to

$$R_2^V(s, m, m, 0) = \frac{m^2}{s} 3 \left\{ 12 \frac{\alpha_s}{\pi} + \frac{629}{6} \left(\frac{\alpha_s}{\pi}\right)^2 + \left(\frac{\alpha_s}{\pi}\right)^3 \left[\frac{89893}{54} - \frac{1645}{36}\pi^2 + \frac{820}{27}\zeta(3) - \frac{36575}{54}\zeta(5) \right] \right\} \tag{32}$$

$$R_2^A(s, m, m, 0) = \frac{m^2}{s} 3 \left\{ -6 - 22 \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[-\frac{2237}{8} + \frac{47}{6}\pi^2 + 97\,\zeta(3) \right] + \left(\frac{\alpha_s}{\pi}\right)^3 \left[-\frac{25024465}{7776} + \frac{15515}{108}\pi^2 + \frac{27545}{12}\zeta(3) + 25\,\zeta(4) - 995\,\zeta(5) \right] \right\} \tag{33}$$

$$R_2^V(s, m, 0, 0) = \frac{m^2}{s} 3 \left\{ -\frac{3}{2} - \frac{5}{2}\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[-\frac{4195}{96} + \frac{47}{24}\pi^2 + \frac{97}{4}\zeta(3) \right] + \left(\frac{\alpha_s}{\pi}\right)^3 \left[-\frac{12079873}{31104} + \frac{2645}{108}\pi^2 + \frac{251185}{432}\zeta(3) + \frac{25}{4}\zeta(4) - \frac{90305}{216}\zeta(5) \right] \right\} \tag{34}$$

or, in the numerical form,

$$R_2^V(s, m, m, 0) = \frac{m^2}{s} 3 \left\{ 12 \frac{\alpha_s}{\pi} + 104.833 \left(\frac{\alpha_s}{\pi}\right)^2 + 547.879 \left(\frac{\alpha_s}{\pi}\right)^3 \right\}, \tag{35}$$

$$R_2^A(s, m, m, 0) = \frac{m^2}{s} 3 \left\{ -6 - 22 \frac{\alpha_s}{\pi} - 85.7136 \left(\frac{\alpha_s}{\pi}\right)^2 - 45.7886 \left(\frac{\alpha_s}{\pi}\right)^3 \right\}, \tag{36}$$

$$R_2^V(s,m,0,0) = R_2^A(s,m,0,0)$$
$$= \frac{m^2}{s} 3 \left\{ -1.5 - 2.5\frac{\alpha_s}{\pi} + 4.7799 \left(\frac{\alpha_s}{\pi}\right)^2 + 125.523 \left(\frac{\alpha_s}{\pi}\right)^3 \right\}. \quad (37)$$

To conclude, we have extended the results of Ref. [3] by evaluating analytically the terms of order $\mathcal{O}(\alpha_s^3 m_q^2)$ for the general case of vector or axial-vector correlator of two non-diagonal quark currents. We have applied the obtained results to find the $\mathcal{O}(\alpha_s^3)$ quadratic quark mass corrections to the hadronic decay rates of W- and Z-bosons. We have shown that to the order in α_s we are dealing with the decay width $\Gamma(W \to \text{hadrons})$ depends on the charmed and bottom quark masses only through a combination $|V_{c,b}|^2(m_c^2+m_b^2)+|V_{u,b}|^2 m_b^2$. Our results can also be applied to determination of the strange quark mass from τ decays. This topic will be treated in a subsequent paper.

This work was supported by BMFT under Contract 057KA92P(0) and INTAS under Contract INTAS-93-0744.

References

[1] A. Blondel, plenary talk presented at the *28th International Conference on High Energy Physics*, 25–31 July 1996, Warsaw, Poland.

[2] K.G. Chetyrkin, J.H. Kuehn and A. Kwiatkowski, Phys. Reports 277(4) (1986) 189.

[3] K.G. Chetyrkin, J.H. Kühn, *Phys. Lett.* **B 248** (1990) 359.

[4] K.G. Chetyrkin, J.H. Kühn, A. Kwiatkowski, *Phys. Lett.* **B 282** (1992) 221.

[5] S.G. Gorishny, A.L. Kataev, S.A. Larin, L.R. Surguladze, *Mod. Phys. Lett.* **A 5** (1990) 2703.

[6] K.G. Chetyrkin and A. Kwiatkowski, *Z. Phys.* **C 59** (1993) 525.

[7] K.G. Chetyrkin, C.A. Dominguez, D. Pirjol and K. Schilcher, *Phys. Rev.* **D 51** (1995) 5090.

[8] L. R. Surguladze, *Phys. Rev.* **D 54** (1996) 2118.

[9] K.G. Chetyrkin, Phys. Lett. B 390 (1997) 309.

[10] O.V. Tarasov, A.A. Vladimirov, A.Yu. Zharkov, *Phys. Lett.* **B 93** (1980) 429.

[11] S.A. Larin and J.A.M. Vermaseren, *Phys. Lett.* **B 303** (1993) 334.

[12] O.V.Tarasov, preprint JINR P2-82-900 (1982).

[13] S.A. Larin, Preprint NIKHEF-H/92-18, hep-ph/9302240 (1992); In Proc. of the Int. Baksan School "Particles and Cosmology" (April 22-27, 1993, Kabardino-Balkaria, Russia) eds. E.N. Alexeev, V.A. Matveev, Kh.S. Nirov, V.A. Rubakov (World Scientific, Singapore, 1994).

[14] D. Broadhurst, *Nucl. Phys.* **B B85** (1975) 189.

[15] S.G. Gorishny, A.L. Kataev, S.A. Larin, *Nouvo Cim.* **92A** (1986) 119.

[16] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, Preprint NIKHEF-H/91-18 (1991).

[17] J.A. M. Vermaseren, Symbolic Manipulation with FORM, Version 2, CAN, Amsterdam, 1991.

[18] S.G. Gorishny, A.L. Kataev and S.A. Larin, *Phys. Lett.* **B 259** (1991) 144.

[19] L.R. Surguladze and M.A. Samuel, *Phys. Rev. Lett.* **66** (1991) 560; erratum ibid, 2416.

[20] K.G. Chetyrkin, Phys. Lett. B 391 (1997) 402.

[21] B.A. Kniehl, J.H. Kühn, Nucl. Phys. B 329 (1990) 547.

[22] B.A. Kniehl, *Phys. Lett.* **B 237** (1990) 127.

[23] K.G. Chetyrkin, *Phys. Lett.* **B 307** (1993) 169.

[24] D.E. Soper and L.R. Surguladze, *Phys. Rev. Lett.* **73** (1994) 2958.

[25] A.H. Hoang, J.H. Kühn and T. Teubner, *Nucl. Phys.* **B 452** (1995) 173.

[26] S.A. Larin, T. van Ritbergen and J.A.M. Vermaseren, Nucl. Phys. B 438 (1995) 278.