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Two-loop QCD Corrections to $b \to c$ Transitions at Zero Recoil: Analytical Results

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Abstract

We present analytical results for the $O(\alpha_s^2)$ contributions to the functions η_A and η_V which parameterize QCD corrections to semileptonic $b \to c$ transitions at zero recoil. Previously obtained approximate results are confirmed. The methods of computing the relevant two-loop diagrams with two mass scales are discussed in some detail.

1 Introduction

The studies of the semileptonic decays of the b–quark provide the best opportunity to determine $|V_{cb}|$, a parameter of the CKM matrix and a fundamental input parameter in the Standard Model. Currently, two different methods of extracting the $|V_{cb}|$ from the experimental data are used. One of them is based on the exclusive decays $B \to D^\star l \nu_l$. In this method the recoil spectrum of the B meson decay is measured:

$$\frac{\mathrm{d}\Gamma(B \to D^\star l \bar{\nu})}{\mathrm{d}w} = f(m_B, m_{D^\star}, w)|V_{cb}|^2 \mathcal{F}^2(w)\left(1 + \frac{\alpha}{\pi}\ln\frac{M_Z}{m_B}\right). \tag{1}$$

In this equation w is the product of the four velocities of the B and $D^\star$ mesons and f is a known function of the masses of the observed particles.

In the limit when the b- and c- quark masses are considered to be large in comparison with Λ_{QCD}, the Heavy Quark Effective Theory [1, 2, 3, 4] can be used to determine the nonperturbative corrections responsible for deviations of $\mathcal{F}(1)$ from unity in a model independent way. For this purpose, small velocity sum rules provide a reliable theoretical framework [5].

From the experimental point of view, the zero recoil point is not directly accessible due to the vanishing phase space. Therefore, one measures $|V_{cb}|^2\ \mathcal{F}^2(w)$ for $w \neq 1$, extrapolates it to $w = 1$ and uses theoretical predictions for $\mathcal{F}(1)$ to extract the value of $|V_{cb}|$.

Apart from the abovementioned nonperturbative effects, $\mathcal{F}(1)$ receives also perturbative QCD corrections. The Lorentz structure of the $b \to c$ decay vertex is $\Gamma_\mu = \gamma_\mu(1-\gamma_5)$. We describe the modifications of the axial and vector parts at zero recoil by two functions:

$$\gamma_\mu \to \eta_V \gamma_\mu, \quad \gamma_\mu\gamma_5 \to \eta_A \gamma_\mu \gamma_5. \tag{2}$$

Both functions η_A and η_V can be expanded in power series in the strong coupling constant:

$$\eta_{A,V} = 1 + \frac{\alpha_s}{\pi} C_F \eta^{(1)}_{A,V} + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \eta^{(2)}_{A,V} + O(\alpha_s^3). \tag{3}$$

The $O(\alpha_s)$ effects have been known for a long time [6, 7, 4]. The $O(\alpha_s^2)$ corrections to zero recoil form factors have been subject of controversy over several years. The need to perform complete two-loop calculation has been emphasized by many authors. The problem was solved in ref. [8], where $O(\alpha_s^2)$ were calculated in the form of a series in $\delta \equiv 1 - m_c/m_b$. The value of $O(\alpha_s^2)$ turned out to be relatively small and therefore the important source of the theoretical uncertainty in the $|V_{cb}|$ determination from exclusive decays has been removed.

The purpose of the present paper is to present analytical results for the $O(\alpha_s^2)$ corrections to two functions η_A and η_V and to discus the methods which have been used for their evaluation. The Feynman diagrams which have to be computed are shown if Fig. 1.

There are several motivations for taking up this issue again. First we want to demonstrate that with the current techniques for doing multiloop calculations the calculation of $O(\alpha_s^2)$

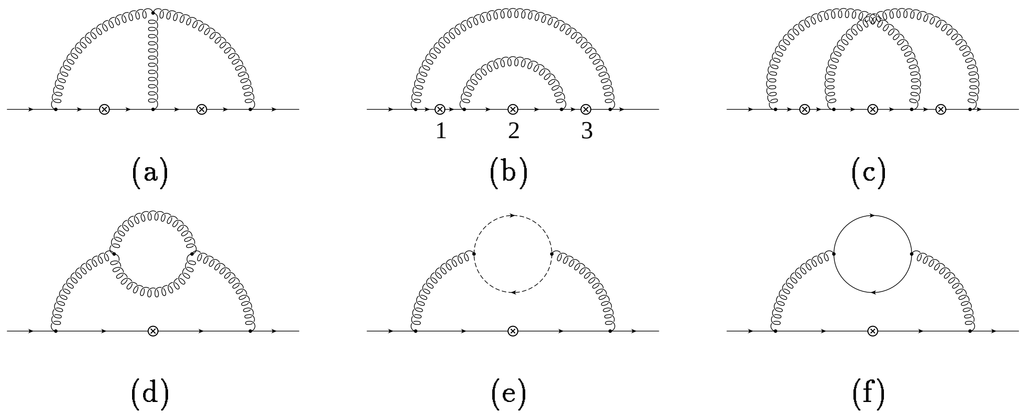


Figure 1: Two-loop QCD corrections to the $b \to c$ transitions at zero recoil. Symbols $\otimes$ mark places where a virtual W boson can possibly couple to the quark line.

correction to $\eta_{A,V}$ is feasible and even relatively simple, if organized properly. Second, having a complete formula one can study it in the whole range of m_c/m_b, while the previously available expansion is not valid for $m_c \ll m_b$. Finally, it is worth reminding that there are almost no results on the decays of fermions neither in QED nor in QCD at the two loop level. As a long standing example, the two loop QED corrections to the muon decay are not known. Being a challenging theoretical problem, the value of $O(\alpha^2)$ corrections to muon decay does not influence precision electroweak physics at present. However, this situation may change if the electroweak parameters, such as $\sin^2\theta_W$, are measured with higher accuracy, e.g. at the future linear collider. In view of the anticipated progress in precision measurements, accumulating as much information as possible about higher order effects in the fermion decays is warranted.

The feasibility of the analytical calculation at zero recoil is mainly connected with the particular kinematical configuration. We denote the momentum of the b–quark in the initial state by p with the on-shell condition $p^2 = m_b^2$. At zero recoil, the c–quark in the final state has the momentum ωp, where $\omega = m_c/m_b$. The virtual W boson, which induces the semileptonic transition, carries off the momentum $(1-\omega)p$. Hence, the zero recoil kinematics dictates that the momenta of the particles in the final state are parallel to the momentum of the particle in the initial state.

To put this in more simple terms, let us consider the zero recoil $b \to c$ transition in the rest frame of the b quark. In this case the c–quark in the final state remains at rest and the energy release $m_b - m_c$ is carried off by the current which induces this transition. From this picture it is evident that no real gluon radiation can appear in the zero recoil transition.

From the technical point of view, the fact that all momenta are parallel to each other simplifies significantly the calculations of the Feynman integrals. The Feynman integrals which appear are similar to those studied in the context of the quark wave function and mass renormalization constants calculation, except that here we have to deal with two (rather than one) mass scales. Below we shall demonstrate that if this similarity is properly exploited, the analytical calculation of the form factors at zero recoil turns out to be feasible and even relatively simple.

The paper is organized as follows. In section 2 we describe in some detail methods of computing two-loop integrals with two mass scales needed in the zero recoil calculation. We pay special attention to the subtraction of infrared divergences so that the difficult parts of the calculation can be done in 4 dimensions. In section 3 the special case of diagrams with a massive fermion loop inserted in a gluon propagator is considered. Our final result is presented and compared with the previous approximations in section 4.

2 Basic scalar integrals

To extract a contribution of a given Feynman diagram to $\eta_{A,V}$ we multiply it by $\bar{u}_c\gamma_\mu u_b$ or $\bar{u}_c\gamma_\mu\gamma_5 u_b$ and average over fermion polarizations. In this way one gets scalar products of four momenta in the numerator which one expresses through the available denominator structures. This leads to a reduction of the number of the propagators for a given Feynman diagram. After this step the remaining Feynman integrals have to be calculated.

Before demonstrating various types of integrals we introduce some notations. We denote the loop momenta by k and l and work in the Euclidean space. There are nine possible propagators which appear in the Feynman integrals is this calculation:

$$S_1 = k^2, \quad S_2 = l^2, \quad S_3 = (k-l)^2, \tag{4}$$

$$S_4 = k^2 + 2pk, \quad S_5 = l^2 + 2pl, \quad S_6 = (k+l)^2 + 2p(k+l),$$
$$S_7 = k^2 + 2\omega pk, \quad S_8 = l^2 + 2\omega pl, \quad S_9 = (k+l)^2 + 2\omega p(k+l).$$

An arbitrary Feynman integral constructed from these objects will be denoted as

$$D(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9|p,\omega) = \int \frac{[\mathrm{d}k][\mathrm{d}l]}{S_1^{\alpha_1} S_2^{\alpha_2} S_3^{\alpha_3} S_4^{\alpha_4} S_5^{\alpha_5} S_6^{\alpha_6} S_7^{\alpha_7} S_8^{\alpha_8} S_9^{\alpha_9}} \tag{5}$$

with

$$[\mathrm{d}k] = \frac{d^D k}{\pi^{D/2}}, \qquad D = 4 - 2\epsilon. \tag{6}$$

If the last two arguments of $D(....|Q,x)$ are shown explicitly they denote objects which should replace p and ω in S_i. Otherwise it is assumed that $Q = p$ and $x = \omega$. We use m_b as a unit of mass. The proper dimension of the integrals can be easily restored, if needed.

For the presentation of our results for the relevant Feynman integrals it is convenient to introduce two auxiliary functions:

$$
\begin{aligned}
R_1 &= \frac{\pi^2}{6} - \mathrm{Li}_2(1-\omega) = \mathrm{Li}_2(\omega) + \ln(1-\omega)\ln(\omega), \\
R_2 &= \mathrm{Li}_2(-\omega) + \ln(1+\omega)\ln(\omega).
\end{aligned} \tag{7}
$$

If the argument of $R_{1,2}$ is given explicitly, it replaces ω in the above definitions.

2.1 Scalar integrals with six propagators

The highest number of propagators in Feynman integrals in the zero recoil calculation is six. Both planar (like in Fig. 1a,b) and non–planar (Fig. 1c) integrals with six propagators appear.

Planar diagrams with six propagators

Let us consider the integral $D(1,0,1,1,1,0,1,1,0)$:

$$
D(1,0,1,1,1,0,1,1,0) = \int \frac{[\mathrm{d}k][\mathrm{d}l]}{k^2\ (l-k)^2(k^2+2pk)(l^2+2pl)(k^2+2\omega pk)(l^2+2\omega pl)}. \tag{8}
$$

The idea which permits the calculation of this integral is the following. Consider the product of S_4 and S_7 or of S_5 and S_8 in the integrand. For both it is possible to perform partial fractioning, like

$$
\frac{1}{(k^2+2pk)(k^2+2\omega pk)} = \frac{1}{1-\omega}\frac{1}{k^2}\left(\frac{1}{k^2+2pk} - \frac{\omega}{k^2+2\omega pk}\right). \tag{9}
$$

For $D(1,0,1,1,1,0,1,1,0)$ we therefore write

$$
\frac{1}{S_4S_7S_5S_8} = \frac{1}{(1-\omega)^2}\frac{1}{S_1S_2}\left(\frac{1}{S_4} - \frac{\omega}{S_7}\right)\left(\frac{1}{S_5} - \frac{\omega}{S_8}\right) \tag{10}
$$

and, accordingly,

$$
\begin{aligned}
D(1,0,1,1,1,0,1,1,0) &= \frac{1}{(1-\omega)^2}\Big[D(2,1,1,1,1,0,0,0,0) + \omega^2 D(2,1,1,0,0,0,1,1,0) \\
&\quad -\omega D(2,1,1,1,0,0,0,1,0) - \omega D(2,1,1,0,1,0,1,0,0)\Big].
\end{aligned} \tag{11}
$$

We notice that the first two D structures in the square brackets are two–loop two–point on–shell integrals with a single mass scale. The only difference between them is in the incoming momenta and in the masses of the particles inside the loop. An algorithm to compute an arbitrary integral of this type is known [9, 10, 11] (the underlying idea is the integration by parts method [12]). A large number of such integrals had to be computed in the approximate approach to the present problem [8]. For that purpose a new implementation of the recurrence algorithm was necessary. The ability to compute such integrals determines our strategy also in the present case.

Apart from the known single scale integrals, eq. (11) contains two unknown integrals. Let us consider one of them as an example:

$$D(2,1,1,1,0,0,0,1,0) = \int \frac{[\mathrm{d}k][\mathrm{d}l]}{(k^2)^2 l^2 (l-k)^2 (k^2+2pk)(l^2+2\omega pl)}.$$

The price we have to pay for partial fractioning is that the integrals we obtain have stronger infrared divergences than the original ones. The integration in D–dimensions is much more complicated than if the number of space–time dimensions could be equated to 4 from the start. In circumventing the problem the following trick proved to be useful. We introduce a Feynman parameter to combine two propagators in the integrand of $D(2,1,1,1,0,0,0,1,0)$:

$$\frac{1}{(k^2)^2(k^2+2pk)} = 2\int\limits_0^1 \frac{\mathrm{d}t\ (1-t)}{(k^2+2pkt)^3}.$$

Introducing the variable $t = \omega\mu$ one gets

$$\int\limits_0^1 \frac{\mathrm{d}t\ (1-t)}{(k^2+2pkt)^3} = \omega \int\limits_0^{1/\omega} \frac{\mathrm{d}\mu(1-\omega\mu)}{(k^2+2pk\omega\mu)^3}.$$

Splitting the integration over μ into two parts: from 0 to 1 and from 1 to $1/\omega$, we get the following representation for $D(2,1,1,1,0,0,0,1,0)$:

$$\begin{aligned} D(2,1,1,1,0,0,0,1,0) \quad = \quad & \omega D(2,1,1,0,0,0,1,1,0) + \omega(1-\omega)D(1,1,1,0,0,0,2,1,0) \\ & + \omega(1-\omega)^2 D(0,1,1,0,0,0,3,1,0) + \Delta, \end{aligned} \tag{12}$$

where Δ is:

$$\Delta = 2\omega \int\limits_1^{1/\omega} \mathrm{d}\mu(1-\omega\mu)\left[D(0,1,1,0,1,0,3,0,0|p\omega,\mu) - \frac{1}{\mu^3}D(0,1,1,0,1,0,3,0,0|p\omega,1)\right].$$

The advantage of this representation is that all the D functions, listed in eq. (12) are now of the single scale type and their values are available. On the other hand, the part of the result denoted by Δ is finite and its calculation can be performed in four dimensions.

Let us stress this point once again. One of the obstacles in calculating the zero recoil form factors is the fact that individual Feynman integrals have infrared divergences, which should disappear in the sum. It is the appearance of these divergences, which makes the calculation tough. We have demonstrated above that in the zero recoil calculation one can extract the infrared divergences in the form of the on–shell integrals which are known.

For the sake of illustration, we write down the following finite result:

$$D(0,1,1,0,1,0,3,0,0|p\omega,\mu) - \frac{1}{\mu^3}D(0,1,1,0,1,0,3,0,0|p\omega,1) = \frac{1}{2\omega^4}\frac{1-\mu+\ln\mu}{(1-\mu)\mu^3}. \quad (13)$$

The simplicity of this result leaves no doubts that the remaining integration over μ can easily be performed. We thus arrive at the result for the planar diagram with six propagators:

$$\begin{aligned} &D(1,1,0,1,1,0,1,1,0) \\ &= -\frac{1}{\omega^2}\left[\frac{12\epsilon^2+2\epsilon-1}{8\epsilon^2} - \frac{1+\omega}{1-\omega}\mathrm{Li}_2(1-\omega) - \frac{(2+\omega)\ln^2(\omega)}{2(1-\omega)} + \frac{(1-2\epsilon)\ln(\omega)}{2\epsilon(1-\omega)} - \frac{13\pi^2}{48}\right]. \end{aligned} \quad (14)$$

This is the only planar diagram with six different propagators which is necessary for the present calculation.

Non-planar diagrams with six propagators

Another integral with 6 propagators we need is $D(1,1,0,1,1,1,0,1,0)$. Performing partial fractioning we get

$$D(1,1,0,1,1,1,0,1,0) = \frac{1}{1-\omega}D(1,2,0,1,1,1,0,0,0) - \frac{\omega}{1-\omega}D(1,2,0,1,0,1,0,1,0).$$

The first of these integrals is of the single scale type. To calculate the second one, we use the same trick as described above and get the following representation:

$$\begin{aligned} D(1,2,0,1,0,1,0,1,0) &= \frac{1}{\omega}D(1,2,0,1,1,1,0,0,0) \\ &-\frac{1-\omega}{\omega^2}D(1,1,0,1,2,1,0,0,0) - \frac{2}{\omega}\int_\omega^1 \frac{dt}{t}\left(1-\frac{t}{\omega}\right)D(0,1,0,3,0,0,0,1,1|t^{-1}). \end{aligned} \quad (15)$$

The first two integrals in this expression are of the single scale type. The expression for the Feynman diagram under the t integration will be presented later (eq. 20). Direct integration gives:

$$\begin{aligned} \int_\omega^1 \frac{dt}{t}\left(1-\frac{t}{\omega}\right) D(0,1,0,3,0,0,0,1,1|t^{-1}) &= -\frac{(1-3\omega)^2}{16\omega^2}R_1 + \frac{(1+\omega)^2}{16\omega^2}R_2 \\ &-\frac{1-\omega}{4\omega}\ln^2(\omega) - \frac{5}{8\omega}\ln(\omega) - \frac{7(1-\omega)}{8\omega} - \frac{\pi^2(1-4\omega)}{48\omega}. \end{aligned} \quad (16)$$

From this an expression for $D(1,1,0,1,1,1,0,1,0)$ can be constructed.

The diagram $D(1,1,0,1,0,1,0,1,1)$ represents a more challenging problem. We note that it has an overall infrared divergence and cannot be calculated in $D=4$. To compute it we introduce the operator

$$\hat{L} = p_\mu \frac{\partial}{\partial p_\mu}.$$

For dimensional reasons the integral we are interested in is proportional to $(p^2)^{-2-2\epsilon}$. By applying the operator $\hat{L}$ to it one finds

$$\hat{L} D(1,1,0,1,0,1,0,1,1) = -(4+4\epsilon)\ D(1,1,0,1,0,1,0,1,1). \tag{17}$$

By applying the same operator to the integrand we get

$$\begin{aligned} -(1+4\epsilon)\ D(1,1,0,1,0,1,0,1,1) &= D(0,1,0,2,0,1,0,1,1) + D(1,0,0,1,0,1,0,2,1) \\ &+ \frac{1}{1-\omega} D(1,1,0,1,0,2,0,1,0) - \frac{\omega}{1-\omega} D(1,1,0,1,0,0,0,1,2). \end{aligned} \tag{18}$$

It is useful to note that Feynman integrals in the above equation satisfy the following symmetry relations

$$D(1,0,0,1,0,1,0,2,1) = \omega^{-4-4\epsilon}\ D(0,1,0,2,0,1,0,1,1|\omega^{-1}),$$

$$D(1,1,0,1,0,0,0,1,2) = \omega^{-4-4\epsilon}\ D(1,1,0,1,0,2,0,1,0|\omega^{-1}).$$

Therefore, it is enough to know two of them.

We present the calculation of $D(0,1,0,2,0,1,0,1,1)$ which turns out to be the most complicated integral. Applying the operator $\hat{L}$ to it once more, we get the following identity:

$$\begin{aligned} -2D(0,1,0,2,0,1,0,1,1) &= \frac{2}{1-\omega} D(0,1,0,3,0,1,0,1,0) \\ &- \frac{2}{1-\omega} D(0,1,0,3,0,0,0,1,1) + \frac{2}{\omega} D(0,1,0,3,0,1,0,0,1) - \frac{2}{\omega} D(0,0,0,3,0,1,0,1,1) \\ &+ \frac{1}{1-\omega} D(0,1,0,2,0,2,0,1,0) + D(0,0,0,2,0,1,0,2,1) - \frac{\omega}{1-\omega} D(0,1,0,2,0,0,0,1,2). \end{aligned} \tag{19}$$

All Feynman integrals in this equation are finite and relatively simple to calculate directly. The results for individual Feynman integrals read:

$$\begin{aligned} D(0,1,0,3,0,0,0,1,1) &= \frac{1}{2\omega}\left[\frac{1}{4}\Big(R_2 - R_1\Big) + \frac{\omega\ln(\omega)}{2(1-\omega)^2} + \frac{1}{2(1-\omega)} + \frac{\pi^2}{8}\right], \qquad (20) \\ D(0,1,0,2,0,0,0,1,2) &= \frac{1}{\omega^2}\left[\frac{1}{4}\Big(R_2 - R_1\Big) - \frac{\omega\ln(\omega)}{2(1-\omega)^2} - \frac{1}{2(1-\omega)} + \frac{\pi^2}{8}\right], \end{aligned}$$

$$D(0,1,0,3,0,1,0,0,1) \;=\; -\frac{1-\omega+\ln(\omega)}{2(1-\omega)^2},$$
$$D(0,0,0,3,0,1,0,1,1) \;=\; \frac{1+\omega}{\omega(1-\omega)}\left[\frac{1}{8}\Big(R_1-R_2\Big)-\frac{\omega\ln(\omega)}{4(1+\omega)^2}-\frac{\pi^2}{16}\frac{\omega}{1+\omega}\right],$$
$$D(0,0,0,2,0,1,0,2,1) \;=\; -\frac{1+\omega}{\omega^2(1-\omega)}\left[\frac{1}{4}\Big(R_2-R_1\Big)-\frac{\omega\ln(\omega)}{2(1+\omega)^2}+\frac{\pi^2}{8}\frac{\omega}{1+\omega}\right],$$

$$2D(0,1,0,3,0,1,0,1,0)+D(0,1,0,2,0,2,0,1,0)=\frac{1}{2\omega}\Big(R_1-R_2\Big).$$

The last missing contribution for the scalar non–planar diagram with six propagators is $D(1,1,0,1,0,2,0,1,0)$. This integral is infrared divergent and therefore its direct evaluation is not convenient. However, the trick described in the context of the planar integral with six propagators is useful also here. One finds the following representation:

$$D(1,1,0,1,0,2,0,1,0)=\frac{1}{\omega}\left[D(1,1,0,1,1,2,0,0,0)-\int\limits_{\omega}^{1}\frac{\mathrm{d}t}{t^4}D(0,1,0,2,0,0,0,1,2|t^{-1})\right].$$

The result for the Feynman integral under the integration over t is presented in the the list of the integrals given above. The final integration over t is elementary. As a result we find

$$\begin{aligned}&D(1,1,0,1,0,1,0,1,1)\\ &\quad=-\frac{1}{4\omega^2\epsilon}+\frac{1}{4\omega^2(1-\omega)}\Big[2(1+\omega)(R_1-R_2)+4\ln(\omega)+12(1-\omega)-\pi^2\Big]. \qquad (21)\end{aligned}$$

2.2 Scalar integrals with five propagators

The scalar integrals with five propagators represent probably the most difficult part of this work. The reason is that these integrals contain trilogarithms, therefore evaluation of the corresponding Feynman integrals is harder and the resulting expressions are rather complicated.

Planar diagrams with five propagators

The most difficult planar diagram is $D(1,1,1,1,0,0,0,1,0)$. To calculate it we use the trick described in the previous section, which allows us to extract on–shell divergent piece from this diagram. The following representation is useful:

$$D(1,1,1,1,0,0,0,1,0)=\frac{1}{\omega}\left[D(1,1,1,1,1,0,0,0,0)-\int\limits_{\omega}^{1}\mathrm{d}t\;D(1,0,1,1,0,0,0,2,0|t)\right],$$

where

$$D(1,0,1,1,0,0,0,2,0|t) = -\frac{2}{t}R_1(t) - \frac{\ln^2(t)}{1-t} + \frac{2\pi^2}{3t}.$$

Performing the integration over t one gets:

$$\begin{aligned} D(1,1,1,1,0,0,0,1,0) &= \frac{1}{\omega}D(1,1,1,1,1,0,0,0,0) - \frac{6\Big(\mathrm{Li}_3(\omega)-\zeta_3\Big)}{\omega} \\ &+ \frac{1}{\omega}\left[4\ln(\omega)\mathrm{Li}_2(\omega) + \ln^2(\omega)\ln(1-\omega) + \frac{2\pi^2}{3}\ln^2(\omega)\right]. \end{aligned} \quad (22)$$

Non-planar diagrams with five propagators

This is the most difficult case. Let us start our discussion with $D(1,1,0,1,0,0,0,1,1)$. We find that this diagram satisfies the following relation:

$$D(1,1,0,1,0,0,0,1,1) = \frac{1}{\omega^2}H\left(\frac{1}{\omega}\right), \quad (23)$$

where the function H is given by

$$H(\omega) = \frac{1}{\omega}\int_0^\omega D(1,0,0,1,0,1,0,2,0|t)\mathrm{d}t.$$

For the expression under the integral we get the following result:

$$D(1,0,0,1,0,1,0,2,0|t) = -\frac{2(1+t)}{t(1-t)}R_2(t) + \frac{\ln^2(t)}{1-t} - \frac{\pi^2}{3(1-t)}. \quad (24)$$

This nice integral representation essentially solves the problem. The subsequent integration over t can be performed; however, the final result is complicated and we relegate it to the appendix. For all practical purposes, like evaluation of the final result as a function of ω, the integral representation provides a more useful starting point than the exact formula.

There are two other integrals which are necessary. We will discuss the calculation of one of them $D(0,1,0,1,0,1,0,1,1)$. For the time being let us denote it as $F(\omega)$.

It turns out to be convenient to derive a differential equation for this function. Differentiating it with respect to ω one gets:

$$\left(-\frac{\mathrm{d}}{\mathrm{d}\omega} - \frac{1-2\omega}{\omega(1-\omega)}\right)F(\omega) = J(\omega), \quad (25)$$

$$J(\omega) = -\frac{1}{\omega}D(0,0,0,1,0,1,0,2,1) + \frac{1}{1-\omega}D(0,1,0,1,0,0,0,1,2).$$

Writing $F(\omega) = \rho(\omega)G(\omega)$ with

$$\rho(\omega) = \frac{1}{\omega(1-\omega)}$$

one gets a new equation for $G(\omega)$:

$$\frac{\mathrm{d}G(\omega)}{\mathrm{d}\omega} = \omega(1-\omega)J(\omega). \tag{26}$$

In order to determine the right hand side of this equation we have to find $J(\omega)$. For this we need two Feynman integrals which can be computed directly

$$\begin{aligned}
D(0,1,0,1,0,0,0,1,2) &= -\frac{2R_1}{\omega} + \frac{2(1+\omega)}{\omega(1-\omega)}R_2 - \frac{2\ln^2(\omega)}{1-\omega} + \frac{\pi^2(2-\omega)}{3\omega(1-\omega)}, \\
D(0,0,0,1,0,1,0,2,1) &= \frac{2R_1}{\omega} - \frac{2R_2}{1-\omega} + \frac{(1+2\omega)\ln^2(\omega)}{1-\omega^2} - \frac{\pi^2(2-\omega)}{3(1-\omega^2)}.
\end{aligned}$$

and

$$J_1(\omega) \equiv \omega(1-\omega)J(\omega) = -\frac{2R_1}{\omega} + \frac{4R_2}{1-\omega} - \frac{(1+3\omega)\ln^2(\omega)}{1-\omega^2} + \frac{2\pi^2(2-\omega)}{3(1-\omega^2)}. \tag{27}$$

With this input the differential equation (26) leads to an integral representation for $F(\omega)$.

$$D(0,1,0,1,0,1,0,1,1) = \frac{1}{\omega(1-\omega)}\int_{\omega}^{1} \mathrm{d}t J_1(t).$$

The analytical result can be found in the appendix.

2.3 Feynman integrals with four propagators

A lot of integrals with four propagators appear in the course of this calculation. A large fraction of them can be calculated directly. In some cases however some special methods appear to be useful.

Below we list some useful results. For the diagram $D(0,0,0,1,0,1,0,1,1)$ we find the following:

$$D(0,0,0,1,0,1,0,1,1) = \frac{(1-\omega^{1-4\epsilon})\,\Gamma(2\epsilon)\,B(1-\epsilon,1-\epsilon)}{\epsilon\,(1-3\epsilon)\,(1-\omega)} - \frac{1}{1-\omega}\left(F_1(\omega) - \omega F_1\left(\frac{1}{\omega}\right)\right),$$

where

$$F_1(\omega) = -\frac{2(1-\omega)^2R_1}{\omega} - \frac{2(1-\omega)R_2}{\omega} + \frac{\omega^2\ln^2(\omega)}{1+\omega} + \frac{\pi^2(1+2\omega^2)}{3(1+\omega)}.$$

For the diagram $D(1,0,0,1,0,1,0,1,0)$ we have:

$$
\begin{aligned}
D(1,0,0,1,0,1,0,1,0) &= \frac{\Gamma(2\epsilon)B(1-\epsilon,1-\epsilon)}{\epsilon(1-3\epsilon)} \\
&\quad -\frac{2(1+\omega)}{\omega}R_1 - \frac{2(1+\omega)^2}{\omega}R_2 - \frac{\omega^2\ln^2(\omega)}{1-\omega} - \frac{\pi^2(1+\omega)}{3}. \qquad (28)
\end{aligned}
$$

Another type of diagrams we have to deal with are those with an internal massive bubble, for example $D(0,1,0,1,0,1,0,1,0)$. Similar integrals were studied in [9] in connection with the investigation of the quark wave function and mass renormalization constants. Methods developed there permit a calculation of the integrals of this type necessary in our case and will be useful also in case of massive quark loop insertion in the gluon propagator (presented in the next section). For example for the diagram $D(0,1,0,1,0,1,0,1,0)$ we find

$$
\begin{aligned}
D(0,1,0,1,0,1,0,1,0) &= D(1,1,1,0,0,0,0,1,0) + \frac{2(1-\omega)^2R_1}{\omega^2} \\
&\quad + \frac{2(1+\omega)^2R_2}{\omega^2} - 2\ln^2(\omega) - \frac{\pi^2}{3}. \qquad (29)
\end{aligned}
$$

Other diagrams which appear in the course of this calculation, like the diagrams with three propagators, can be reduced to the diagrams with four propagators similar to the ones discussed above but with the different structures in the numerator. All of these diagrams can be studied in a similar manner and we do not present them here in any detail.

The results presented above are sufficient to get the analytical result for the major part of the diagrams involved in this calculation. However, they are not sufficient for the diagrams with the internal massive fermion loop. We shall discuss their evaluation in the next section.

3 Diagrams with a massive fermion loop

The diagrams in Fig. 1 with a massive b or c quark loop insertion in the gluon propagator represent a somewhat special case. It is clear how to extract their divergences. The ultraviolet divergences are canceled by wave function renormalization. On the other hand, these diagrams give infrared finite contributions if we use on–shell renormalization for the coupling constant. Therefore, in order to deal with the finite contributions from the very beginning one should study the correction to the vertex together with the wave function renormalization and renormalize the coupling constant in a QED–like way.

The contribution of the massive fermions to the gluon polarization function $\Pi(k^2)$ can be written in a form of a dispersion integral (for the purpose of illustration we consider a particle in the internal loop to be b–quark) subtracted at $k^2 = 0$. After that, the result appears as a convolution of a one-loop correction evaluated with the gluon of mass λ with the fermion spectral density. The following transformation rule is valid:

$$\frac{g_{\mu\nu}}{k^2} \to g_{\mu\nu} \frac{\alpha_s C_F T_R}{3\pi} \int\limits_{4m_b^2}^{\infty} \frac{d\lambda^2}{\lambda^2(k^2-\lambda^2)} \left(1+\frac{2m_b^2}{\lambda^2}\right) \sqrt{1-\frac{4m_b^2}{\lambda^2}}. \tag{30}$$

To proceed further, we first calculate the one-loop form factors $\eta_{A,V}$ with a massive gluon. Afterwards the result must be integrated over the mass of the gluon with the fermion spectral density:

$$\eta_{A,V}(b) \sim \int\limits_{4m_b^2}^{\infty} \frac{d\lambda^2}{\lambda^2} \left(1+\frac{2m_b^2}{\lambda^2}\right) \sqrt{1-\frac{4m_b^2}{\lambda^2}}\, \eta_{A,V}^{(1,\lambda)}. \tag{31}$$

Here $\eta_{A,V}(b)$ denotes the contributions of diagrams with a b quark loop inserted in the gluon propagator and $\eta_{A,V}^{(1,\lambda)}$ denotes the one–loop form factors calculated with a massive gluon propagator $1/(k^2-\lambda^2)$.

It is advantageous to write the one–loop result leaving the integration over the last of the Feynman parameters untouched. Namely, both one–loop form factors calculated with the finite mass of the gluon $\eta_{A,V}^{(1,\lambda)}$ can be written as:

$$\eta_{A,V}^{(1,\lambda)} = \int\limits_0^1 d\beta \left(\frac{P_1^{A,V}(\beta,\omega)}{\lambda^2\beta+(1-\beta)^2} - \frac{P_2^{A,V}(\beta,\omega)}{\lambda^2\beta+\omega^2(1-\beta)^2} \right) \tag{32}$$

where

$$P_1^A(\beta,\omega) = -\frac{\omega(2-29\beta+51\beta\omega+34\beta^2-30\beta^2\omega-7\beta^3+9\beta^3\omega+18\omega)}{6(1-\omega)},$$

$$P_2^A(\beta,\omega) = -\frac{\omega^3(30\beta^2-9\beta^3+7\beta^3\omega-34\beta^2\omega-2\omega-51\beta+29\beta\omega-18)}{6(1-\omega)},$$

$$P_1^V(\beta,\omega) = -\frac{\omega(-2-7\beta+17\beta\omega+14\beta^2-10\beta^2\omega-5\beta^3+3\beta^3\omega+6\omega)}{2(1-\omega)},$$

$$P_2^V(\beta,\omega) = -\frac{\omega^3(-6+2\omega-17\beta+10\beta^2-3\beta^3+7\beta\omega-14\beta^2\omega+5\beta^3\omega)}{2(1-\omega)}.$$

Substituting this result to eq. (31) and performing the integration over λ one gets

$$\eta_{A,V}(b) \sim \int\limits_0^1 d\beta \left[\frac{P_1^{A,V}(\beta,\omega)}{(1-\beta)^2}\ \Pi\left(\frac{(1-\beta)^2}{\beta}\right) - \frac{P_2^{A,V}(\beta,\omega)}{\omega^2(1-\beta)^2}\ \Pi\left(\frac{\omega^2(1-\beta)^2}{\beta}\right) \right],$$

where $\Pi(s)$ is the vacuum polarization function:

$$\Pi(s) = \left(1-\frac{2}{s}\right) v \ln\left(\frac{v+1}{v-1}\right) + \frac{4}{s} - \frac{5}{3} \tag{33}$$

with

$$v=\sqrt{1+\frac{4}{s}}.$$

Subsequent calculation of the one-parametric integrals is "tedious but straightforward."

So far we have discussed the calculation of the massive fermion bubbles in what can be called on-shell or QED like normalization. The final results which we would like to get are the ones with the $\overline{MS}$–coupling constant α_s renormalized at the symmetric point $\mu=\sqrt{m_b m_c}$. The transformation to this form amounts to performing a scale transformation in the coupling constant. Finally, for the b–quark bubble contribution to the zero recoil form factors one gets:

$$\eta_A(b)=\frac{(3\omega^4-7\omega^3-7\omega^2-25\omega+36)R_1}{48\omega^4}+\frac{(5\omega-18\omega^2-34\omega^4-21\omega^5+36)R_2}{48\omega^4(1-\omega)}$$

$$-\frac{(1+\omega)\ln^2(\omega)}{4(1-\omega)}-\frac{(15-4\omega+18\omega^2-37\omega^3)\ln(\omega)}{24(1-\omega)\omega^2}+\frac{72-33\omega+104\omega^2-143\omega^3}{72\omega^2(1-\omega)}-\frac{\pi^2(5-3\omega)}{36(1-\omega)}$$

and

$$\eta_V(b)=\frac{(\omega^4+3\omega^3+3\omega^2-19\omega+12)R_1}{16\omega^4}+\frac{(-7\omega^5+12-22\omega^2-9\omega-6\omega^4)R_2}{16\omega^4(1-\omega)}$$

$$-\frac{(1+\omega)\ln^2(\omega)}{4(1-\omega)}-\frac{(15-36\omega+14\omega^2-25\omega^3)\ln(\omega)}{24\omega^2(1-\omega)}+\frac{24-51\omega+62\omega^2-35\omega^3}{24\omega^2(1-\omega)}-\frac{\pi^2(3-\omega)}{12(1-\omega)}.$$

4 Results

It is convenient to divide up the functions $\eta^{(2)}_{A,V}$ into parts proportional to various SU(3) factors (an overall factor C_F has been factored out):

$$\eta^{(2)}_{A,V}=C_F\eta^F_{A,V}+(C_A-2C_F)\eta^{AF}_{A,V}+T_R N_L\eta^L_{A,V}+T_R\eta^H_{A,V}. \tag{34}$$

For a general SU(N) group $C_A=N$; $C_F=(N^2-1)/2N$; $T_R=1/2$. N_L denotes the number of the light quark flavors whose masses can be neglected. The last term contains contributions of the massive quark loops, with b and c quarks. We neglect the top quark; its impact is suppressed by a factor $\sim m_b^2/m_t^2$. We use the pole masses of b and c quarks and express our results in terms of $\alpha_{\overline{MS}}(\sqrt{m_b m_c})$.

Among the eight coefficient functions in eq. (34) $\eta^L_{A,V}$ are already known exactly [13]. They corresponds to our diagram (f) in fig. 1 with a massless fermion in the loop; they read

$$\begin{aligned}\eta^L_V &= \frac{1}{24}\left[\frac{1+\omega}{1-\omega}\ln(\omega)+2\right],\\ \eta^L_A &= \frac{5}{24}\left[\frac{1+\omega}{1-\omega}\ln(\omega)+\frac{44}{15}\right].\end{aligned} \tag{35}$$

The contribution of the massive quark loops is obtained by combining the result for the b quark loop, obtained in the previous section, with an analogous formula for the c quark. The latter is obtained from the former by a simple symmetry transformation. We find

$$\begin{aligned}
\eta_V^H &= (1-\omega)^3(1+\omega)\frac{12+5\omega+13\omega^2+5\omega^3+12\omega^4}{16\omega^4}R_1 \\
&+(1+\omega)\frac{12-21\omega-\omega^2+\omega^3-14\omega^4+\omega^5-\omega^6-21\omega^7+12\omega^8}{16(1-\omega)\omega^4}R_2 \\
&+\ln^2(\omega)\frac{1+\omega+5\omega^4-3\omega^5}{4(1-\omega)}-\ln(\omega)(1+\omega)\frac{15-51\omega+40\omega^2-51\omega^3+15\omega^4}{24(1-\omega)\omega^2} \\
&-\frac{37-30\omega+66\omega^3-53\omega^4+12\omega^5}{96(1-\omega)}\pi^2+\frac{24-27\omega+70\omega^2-27\omega^3+24\omega^4}{24\omega^2}, \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\eta_A^H &= (1-\omega^2)\frac{36-25\omega+29\omega^2-32\omega^3+29\omega^4-25\omega^5+36\omega^6}{48\omega^4}R_1 \\
&+(1+\omega)\frac{36-31\omega+13\omega^2-13\omega^3-42\omega^4-13\omega^5+13\omega^6-31\omega^7+36\omega^8}{48(1-\omega)\omega^4}R_2 \\
&+\ln^2(\omega)\frac{3+3\omega+7\omega^4-9\omega^5}{12(1-\omega)}-\ln(\omega)(1+\omega)\frac{15-19\omega-19\omega^3+15\omega^4}{24(1-\omega)\omega^2} \\
&+\frac{72+39\omega+286\omega^2+39\omega^3+72\omega^4}{72\omega^2}-\frac{79-10\omega+54\omega^3-127\omega^4+36\omega^5}{288(1-\omega)}\pi^2. \qquad (37)
\end{aligned}$$

The functions $R_{1,2}$ are defined in eq. (7).

Diagrams without fermion loops give rise to two color structures and we chose the combinations C_F^2 and $C_AC_F-2C_F^2$ to describe them. For the functions $\eta_{V,A}^F$ we find

$$\begin{aligned}
\eta_V^F &= -\frac{\omega^2(2+\omega)}{8(1-\omega)^2}\ln^2(\omega)+\left(\frac{89}{96}+\frac{\pi^2}{6}\right)\frac{1+\omega}{1-\omega}\ln(\omega)+\frac{(1+\omega)^3}{4\omega(1-\omega)}\mathrm{Li}_2(1-\omega) \\
&-\frac{1-10\omega+\omega^2}{24\omega}\pi^2+\frac{53}{48}+\frac{3-10\omega+3\omega^2}{(1-\omega)^2}f(\omega), \qquad (38)
\end{aligned}$$

$$\begin{aligned}
\eta_A^F &= \frac{\omega^2(6-\omega)}{24(1-\omega)^2}\ln^2(\omega)+\left(\frac{53}{96}+\frac{\pi^2}{6}\right)\frac{1+\omega}{1-\omega}\ln(\omega)+\frac{(1+\omega)(1-6\omega+\omega^2)}{12\omega(1-\omega)}\mathrm{Li}_2(1-\omega) \\
&-\frac{1-38\omega+\omega^2}{72\omega}\pi^2-\frac{95}{72}+\frac{9-14\omega+9\omega^2}{3(1-\omega)^2}f(\omega), \qquad (39)
\end{aligned}$$

$$f(\omega)=\left(\mathrm{Li}_2(1-\omega)-\frac{\pi^2}{4}\right)\ln(\omega)+\frac{3}{2}\left(\mathrm{Li}_3(\omega)-\zeta_3\right)+\frac{3}{4}\left(\ln(1-\omega)+\frac{1}{8}\right)\ln^2(\omega).$$

The most complicated and difficult to compute are the functions $\eta_{V,A}^{AF}$. We parameterize them using, in addition to the notations introduced before, in terms of four functions $f_{1\ldots4}(\omega)$ for which closed formulas are given in the appendix. In the following formulas we drop the argument ω of f_i:

$$\eta_V^{AF} = \frac{f_1\omega}{8}(1-3\omega) - f_2(1+\omega)\frac{1+2\omega-\omega^2}{16\omega} - f_3\frac{1-\omega+5\omega^2-\omega^3}{16(1-\omega)} + f_4\frac{\omega(1+\omega)}{8(1-\omega)}$$
$$-\frac{(1-\omega^2)(3R_1+R_2)}{16\omega} - R_1\ln(\omega) + \frac{3}{2}\left(\mathrm{Li}_3(\omega)-\zeta_3\right) + \frac{3}{4}\ln(1-\omega)\ln^2(\omega) - \frac{1}{8}\omega\ln^2(\omega)$$
$$-\frac{17}{96}\ln(\omega)\frac{1+\omega}{1-\omega} - \ln(\omega)\frac{(1+\omega)^2\pi^2}{24(1-\omega)\omega} + \frac{5}{96}(1-\omega)\pi^2 - \frac{17}{48}, \tag{40}$$

$$\eta_A^{AF} = -R_1\ln(\omega) + \frac{3}{2}\mathrm{Li}_3(\omega) + \frac{(1-\omega^2)(8-\omega+8\omega^2)}{48\omega^2}R_1$$
$$+\frac{(1+\omega)(8-3\omega+22\omega^2-3\omega^3+8\omega^4)}{48(1-\omega)\omega^2}R_2 + \frac{3}{4}\ln(1-\omega)\ln^2(\omega) - \frac{f_1\omega}{24}(1+9\omega)$$
$$-f_2\frac{3+\omega+19\omega^2-3\omega^3}{48\omega} - f_3\frac{3-11\omega+31\omega^2-3\omega^3}{48(1-\omega)} + f_4\frac{\omega(7+3\omega)}{24(1-\omega)}$$
$$-\ln^2(\omega)\omega\frac{9-6\omega+5\omega^2-4\omega^3}{24(1-\omega)^2} - \pi^2\ln(\omega)\frac{3-2\omega+3\omega^2}{72(1-\omega)\omega} - \frac{61}{96}\ln(\omega)\frac{1+\omega}{1-\omega}$$
$$+\frac{63-46\omega+23\omega^2-8\omega^3}{288(1-\omega)}\pi^2 - \frac{151}{72} - \frac{\pi^2\ln 2}{6} - \frac{5}{4}\zeta_3. \tag{41}$$

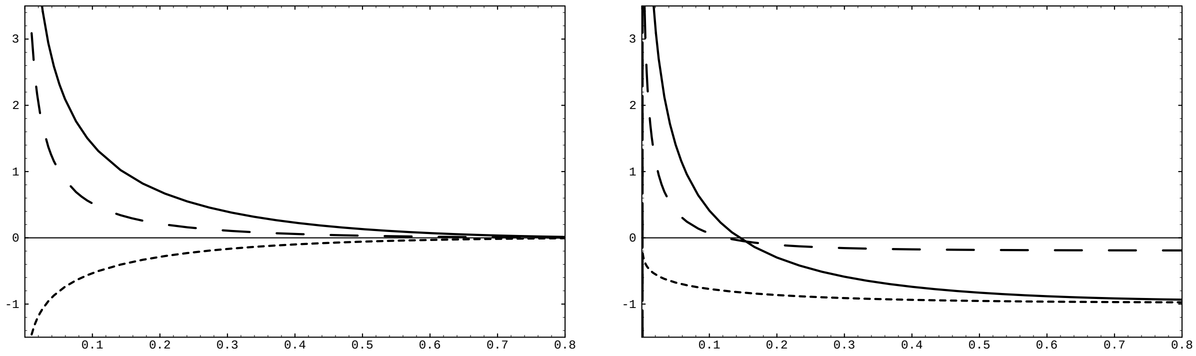
The behavior of the above coefficient functions is plotted in Fig. 2. It can be seen that the corrections to the vector coupling vanish when $\omega \to 1$, in accordance with vector current conservation. For $\omega = 0.3$, which roughly corresponds to the physical value of the mass ratios of c and b quarks, we find the following results

$$\begin{aligned} &\eta_V^H = 0.107667, \quad \eta_V^F = 0.393822, \quad \eta_V^{AF} = -0.167695, \\ &\eta_A^H = -0.154818, \quad \eta_A^F = -0.586889, \quad \eta_A^{AF} = -0.909398, \end{aligned} \tag{42}$$

in excellent agreement with the approximate values found in [8] (see eq. (11) in that reference).

From the above exact formulas we can also find the expansions of all the coefficient functions around $\omega = 0$ which corresponds to the vanishing c quark mass. We give below the terms which do not vanish as $\omega \to 0$:

$$\begin{aligned} \eta_V^H &\to \frac{\ln^2(\omega)}{4} + \frac{13\ln(\omega)}{24} + \frac{1147}{288} - \frac{37\pi^2}{96}, \\ \eta_A^H &\to \frac{\ln^2(\omega)}{4} + \frac{7\ln(\omega)}{8} + \frac{971}{288} - \frac{79\pi^2}{288}, \\ \eta_V^F &\to \frac{9\ln^2(\omega)}{32} + \left(\frac{113}{96} - \frac{\pi^2}{12}\right)\ln(\omega) + \frac{41}{48} + \frac{7\pi^2}{12} - \frac{9\zeta_3}{2}, \\ \eta_A^F &\to \frac{9\ln^2(\omega)}{32} + \left(\frac{61}{96} - \frac{\pi^2}{12}\right)\ln(\omega) - \frac{101}{72} + \frac{17\pi^2}{36} - \frac{9\zeta_3}{2}, \end{aligned}$$



(a) (b)

Figure 2: Coefficient functions η^F (solid), η^H (dashed), η^{AF} (dotted), for the vector (a) and axial case (b), plotted against $\omega = 0..0.8$; none of these functions varies noticably between $\omega = 0.8$ and $\omega = 1$.

$$
\begin{aligned}
\eta_V^{AF} &\to \left(-\frac{5}{96}+\frac{\pi^2}{24}\right)\ln(\omega)-\frac{23}{48}-\frac{\pi^2}{96}+\frac{\pi^2\,\ln(2)}{2}-\frac{9\,\zeta_3}{4}, \\
\eta_A^{AF} &\to \left(-\frac{49}{96}+\frac{\pi^2}{24}\right)\ln(\omega)-\frac{83}{36}+\frac{5\,\pi^2}{32}+\frac{\pi^2\,\ln(2)}{3}-2\,\zeta_3.
\end{aligned}
\tag{43}
$$

5 Conclusions

In this paper we have presented an analytical calculation of the $\mathcal{O}(\alpha_s^2)$ corrections to the functions η_A and η_V which parameterize semileptonic $b \to c$ transition at zero recoil. Numerically, our results are in excellent agreement with the results of the ref. [8] obtained using an approximate approach to the present problem.

This calculation was facilitated by an efficient method of computing two-loop vertex functions with two mass scales in the zero recoil limit which we demonstrated in some detail. Infrared singularities of individual diagrams were shown to be the same as in certain combinations of single scale integrals, for which an exact algorithm is known. We presented various useful tricks which permit a calculation of the remaining two-scale integrals. Since the previous approximate formulas [8] were valid only for similar masses of initial and final quarks, the exact formulas we obtained significantly extend our knowledge of two-loop corrections to fermion decays.

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A Analytical results for the functions f_i

In eq. (40-41) we presented the results for the coefficients η^{AF} in terms of four functions $f_{1\ldots4}$. In this appendix we present closed formulas for f_1 and f_4. The remaining two functions $f_{2,3}$ are obtained from these results by applying the following symmetry relations:

$$f_2(\omega) = \frac{1}{\omega^2} f_1\left(\frac{1}{\omega}\right),$$

$$f_3(\omega) = \frac{1}{\omega^2} f_4\left(\frac{1}{\omega}\right).$$

For $f_1(\omega) \equiv D(0,1,0,1,0,1,0,1,1)$ we find

$$\frac{1}{\omega(1-\omega)}\Bigg[4\mathrm{Li}_3(\omega) + 8\mathrm{Li}_3\left(\frac{1}{1+\omega}\right) + 4\mathrm{Li}_3(1-\omega) + 6\mathrm{Li}_3(-\omega) - 2\mathrm{Li}_3(1-\omega^2)$$
$$-4\mathrm{Li}_3\left(\frac{1+\omega}{2}\right) + 4\mathrm{Li}_3\left(\frac{2\omega}{1+\omega}\right) - \frac{13}{2}\zeta_3 - 4\mathrm{Li}_2\left(\frac{2\omega}{1+\omega}\right)\ln(\omega) - 6\ln(\omega)\mathrm{Li}_2(-\omega) - 2\ln(\omega)\mathrm{Li}_2(\omega)$$
$$+2\ln(2)\ln^2(1+\omega) - 2\ln^2(2)\ln(1+\omega) - 2\ln^2(\omega)\ln(1-\omega) - \ln^2(\omega)\ln(1+\omega)$$
$$+\pi^2\ln(1+\omega) - \frac{\pi^2}{3}\ln(2) + \frac{2}{3}\ln^3(2) - 2\ln^3(1+\omega)\Bigg].$$

The result for $f_4(\omega) \equiv D(1,1,0,1,0,1,0,1,0)$ is

$$\frac{1}{\omega}\Bigg[-2\mathrm{Li}_3(1-\omega^2) - 4\mathrm{Li}_3\left(\frac{1+\omega}{2}\right) + 4\mathrm{Li}_3(1-\omega) + 8\mathrm{Li}_3\left(\frac{1}{1+\omega}\right) - 2\mathrm{Li}_3(\omega)$$
$$+4\mathrm{Li}_3\left(\frac{2\omega}{1+\omega}\right) - \frac{13}{2}\zeta_3 - 4\mathrm{Li}_2\left(\frac{2\omega}{1+\omega}\right)\ln(\omega) + 2\ln(\omega)\mathrm{Li}_2(\omega) - 2\ln(\omega)\mathrm{Li}_2(-\omega)$$
$$+2\ln(2)\ln^2(1+\omega) - \frac{\pi^2}{3}\ln(2) + 2\pi^2\ln(1+\omega) - \ln^2(\omega)\ln(1-\omega) + \frac{2}{3}\ln^3(2)$$
$$-2\ln^3(1+\omega) - 2\ln^2(2)\ln(1+\omega)\Bigg].$$

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