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QCD based Calculation of the semi-inclusive Decay $\eta_Q \to \gamma$ + light Hadrons

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Abstract

A QCD based calculation of the photon spectrum in semi-inclusive η_Q decays is performed. The method applied is an effective theory based on a $1/m_Q$ expansion of QCD.

1 Introduction

The expansion of QCD in inverse powers of the mass has become an indispensable tool for the analysis of heavy hadron decays. This method, formulated as an effective field theory known as Heavy Quark Effective Theory (HQET) [1], has put the physics of heavy quarks on a QCD based and thus model independent framework.

While applications of these methods to systems with a single heavy quark has been studied extensively (see [2] for reviews), there is a lot of motivation to extend this ideas to systems with two or more heavy quarks, because heavy quarkonia (the ψ's and the Υ's) have been known since some time and the observation of flavored "doubly heavy" systems (such as the B_c family of states) is very likely in the near future.

The starting point of the construction of an effective theory for these systems is again the expansion of the relevant Greens functions in $1/m_Q$, where m_Q is the mass of the heavy quark. However, it turns out [3] that one can not use the static limit for two heavy quarks, in particular if their velocities differ only by an amount of order $1/m_Q$, i.e. $vv' - 1 \sim \Lambda_{QCD}/m_Q$ as it is the case in a particle containing two heavy quarks. The static limit breaks down and one is forced to include at least the kinetic energy into the leading order dynamics.

However, unlike the case of one heavy quark an effective theory for heavy quarkonia in terms of the quark and gluon degrees of freedom is more complicated. This is mainly due to the fact that in a heavy quarkonium not only the scale of the light degrees of freedom (as e.g. $\bar{\Lambda}$ in HQET) appears as a scale small compared to the heavy quark mass, but also scales which do depend on the heavy quark mass. This point becomes clear if one studies a system like the positronium first.

In the QED case one may safely assume that the coupling α does not run when one studies a QED bound state consisting of two heavy fermions, which we shall assume to be of the same mass m_Q. The coupling α is a small parameter and hence the inverse Bohr radius $m_Q\alpha$ of such a system is small compared to the heavy mass m_Q. Furthermore, in the case of QED it makes sense to talk about the binding energy E_B of this system, which is again smaller by a factor α, $E_B \sim m_Q\alpha^2$.

Thus in a coulombic system there are two scales, the inverse Bohr radius and the binding energy which are both small compared to the heavy mass:

$$m_Q\alpha^2 \ll m_Q\alpha \ll m_Q. \tag{1}$$

In addition, both scales depend on the heavy mass, in this simple example in a trivial way. The expansion parameter in this case is the relative velocity v_{rel} between the two heavy constituents, and a consistent scheme for the velocity counting has been set up. The velocity in a coulombic system turns out to be $v_{rel} = \alpha$ and hence is indeed a small parameter to expand in. In particular, the binding energy is of the order v_{rel}^2.

Switching now to full QCD the coupling starts to run, introducing another scale Λ_{QCD}. The problem in applying the coulombic hierarchy of scales to realistic quarkonia is related to the question at which point in the hierarchy (1) Λ_{QCD} enters. This is obviously a question how large m_Q is, and for "superheavy" quarkonia one can have the scenario $\Lambda_{QCD} \ll m_Q \alpha^2$ such that the binding is coulombic and – from the point of view of QCD – is of perturbative nature, since $\alpha_s(E_B) \ll 1$. This case is governed by Non-Relativistic QCD (NRQCD) as formulated by Bodwin, Braaten and Lepage [4], where the velocity counting of the QED analogue is used. In fact, this type of counting is also correct as long as the binding is due to instantaneous gluons such that the emission of a dynamical, transverse gluon is suppressed by some power of the velocity.

Realistic quarkonia, such as charmonium of bottomonium, do not seem to fulfill the requirement of being coulombic. This becomes obvious from the spectra as well as from the leptonic widths, which are related to the wave functions at the origin in the simplest picture. Here it seems that Λ_{QCD} is at least larger than the binding energy or maybe even larger than the inverse Bohr radius. In such a case it is not obvious what the parameter is in which an expansion should be formulated. The $1/m_Q$ expansion yields in the numerators combinations of covariant derivatives, some of which may be identified with the relative velocity. In a "minimalistic" approach (called HQ$\overline{\text{Q}}$ET in [5]), all derivatives are kept at the price of a proliferation of unknown matrix elements. In other words, in HQ$\overline{\text{Q}}$ET simply the powers of the operators are counted and one can translate NRQCD into HQ$\overline{\text{Q}}$ET and vice versa, finding that some matrix elements in HQ$\overline{\text{Q}}$ET at leading order are down by factors of v_{rel} in the counting scheme of NRQCD.

In the present paper we shall study a simple process which could shed some light on the question of how to correctly count the "powers" of a matrix element in the $1/m_Q$ expansion for realistic heavy quarkonia. We shall discuss the decay $\eta_Q \to \gamma$ + *light hadrons*, in which the hard subprocess is the annihilation of two heavy quarks into a photon and a gluon, which implies that the two heavy quarks have to be in a color octet state at the matching scale m_Q. The matching to the effective theory thus yields (at the matching scale $\mu = m_Q$) four-quark operators in which the quark-antiquark pair is in a color octet state, and hence in naive factorization these would vanish. Also in NRQCD these matrix elements would require at least an additional dynamical gluon, the emission of which is suppressed by factors of v_{rel}; hence this process is an interesting laboratory to test the size of these octet contributions which seem to play some role in quarkonia production.

In the next section we give a brief review of the effective theory approach used here. In section 3 we set up the necessary machinery for these decays in which the two heavy quarks annihilate and review the short distance contributions. In section 4 we discuss the hadronic matrix elements and the restrictions imposed by heavy quarkonia spin symmetry. In section 5 we apply this method to calculate the total rate and the photon spectrum in $\eta_Q \to \gamma +$ light hadrons. Finally we discuss our results and conclude.

2 Structure of the Effective Theory for Quarkonia

As we shall exploit the fact that the mass of the heavy quark is large compared to both Λ_{QCD} and all other scales such as the analoga of the inverse Bohr radius and the binding energy, we are starting from a $1/m_Q$ expansion of the QCD Lagrangian and the corresponding expansion of the fields up to order $1/m_Q^2$. It is well known that there are different ways to perform such an expansion (as e.g. by integrating out the small component fields from the QCD functional [6] or by a Foldy–Wouthuysen Transformation [7]); although the $1/m_Q$ expansions of their Lagrangians and of their fields look different, the result for the Greens functions will be the same. Hence one may pick the most convenient representation for the application under discussion and we pick [7]

$$\begin{aligned}
\mathcal{L} &= \bar{h}_v^{(+)}(iv\cdot D)h_v^{(+)} + K_1^{(+)} + M_1^{(+)} + K_2^{(+)} + M_2^{(+)} + \mathcal{L}_{glue} + \cdots\,, \\
Q_v^{(+)}(x) &= e^{-im_Q v\cdot x}\left[1 + \frac{1}{2m_Q}(i\not{D}_\perp) + \frac{1}{4m_Q^2}\left(v\cdot D\not{D}_\perp - \frac{1}{2}\not{D}_\perp^2\right) + \cdots\right] h_v^{(+)}(x)\,,
\end{aligned} \tag{2}$$

where v is the velocity of the quarkonia, and the transverse components of the derivative is given by $D_\mu^\perp = (g_{\mu\nu} - v_\mu v_\nu)D^\nu$. The K_i and M_i are operators of higher dimension

$$\begin{aligned}
K_1^{(+)} &= \bar{h}_v^{(+)}\frac{(iD_\perp)^2}{2m_Q}h_v^{(+)}\,, \quad M_1^{(+)} = \frac{1}{2m_Q}\bar{h}_v^{(+)}(-i\sigma_{\mu\nu})(iD_\perp^\mu)(iD_\perp^\nu)h_v^{(+)}\,, \\
K_2^{(+)} &= \frac{1}{8m_Q^2}\bar{h}_v^{(+)}[(iD_\perp^\mu),[(-ivD),(iD^\perp_\mu)]]h_v^{(+)}\,, \\
M_2^{(+)} &= \frac{1}{8m_Q^2}\bar{h}_v^{(+)}(-i\sigma_{\mu\nu})\{(iD_\perp^\mu),[(-ivD),(iD_\perp^\nu)]\}h_v^{(+)}\,, \\
\mathcal{L}_{glue} &= \left(\frac{1}{2m}\right)^2\frac{\alpha_s}{30\pi m^2}\ \mathrm{Tr}\{[iD_\mu, G^{\mu\nu}][iD^\lambda, G_{\lambda\nu}]\}
\end{aligned} \tag{3}$$

$$+ \left(\frac{1}{2m}\right)^2 \frac{i\alpha_s g_s}{360\pi m^2} \, \mathrm{Tr}\{G^{\mu\nu}[G_{\nu\rho}, G^{\rho}{}_{\mu}]\} .$$

Note that at order $1/m_Q^2$ one has also a purely gluonic contribution due to closed heavy quark loops. It contains the gluonic field strength tensor $G_{\mu\nu}$, which is defined by $ig_s G_{\mu\nu} = [D_\mu, D_\nu]$.

The first terms of these two expansions (2) define the static limit, which has been successfully applied to systems with a single heavy quark. In order to describe a system with more than one heavy (anti)quark one has to write down the same expansion (2) for each heavy quark. However, there is no static limit for a system with two heavy quarks if the two heavy quarks move with almost the same velocity as it is the case for a quarkonium; one runs into problems with diverging phases and "complex anomalous dimensions", which are considered in detail in [3].

In order to cure this problem one has to choose the unperturbed system such that these phases are already generated by the leading order dynamics, i.e. instead of the static limit one has to use the non-relativistic Lagrangian. In this case the "power counting" has to be modified in such a way that the leading term is the static plus the kinetic energy term, leading to the fact that the "time derivative" ivD has to be counted as two powers of the "spatial derivative" $iD^\perp$. In physical terms, for a bound state this corresponds to the balance between potential and kinetic energy, since the covariant time derivative ivD contains also the potential.

For a system with a heavy quark and a heavy antiquark one then starts from

$$\mathcal{L}_0 = \bar{h}_v^{(+)}(ivD)h_v^{(+)} - \bar{h}_v^{(-)}(ivD)h_v^{(-)} + \bar{h}_v^{(+)}\frac{(iD_\perp)^2}{2m_Q}h_v^{(+)} + \bar{h}_v^{(-)}\frac{(iD_\perp)^2}{2m_Q}h_v^{(-)} \quad (4)$$

where we have assumed for simplicity that the two quarks have the same mass; the case of unequal mass is obvious.

Naive power counting would suggest to include also the Pauli term $\vec{\sigma}\cdot\vec{B}$ into the leading-order Lagrangian. However, in a coulombic system this term is in fact down by powers of v_{rel}, since the chromomagnetic field is generated by the relative motion of the two heavy quarks; this has been made manifest in [8].

Most of the success of HQET is due to heavy quark flavor and spin symmetry. However, once one uses (4) the symmetries are somewhat different for HQ$\overline{\text{Q}}$ET or NRQCD. First of all, (4) depends on the mass through the kinetic energy term; consequently the states will depend on m_Q in a non-perturbative way and heavy flavor symmetry is lost. In this way the mass dependent small scales (the inverse Bohr radius and the binding energy) are generated by taking matrix elements of operators involving derivatives with these states. On the other hand, since the chromomagnetic moment operator does not appear to leading order, (4) does not depend on the spins of the two heavy quarks, and hence there is a spin symmetry which is larger than in HQET because we have two heavy quark spins; the resulting symmetry is an $SU(2) \otimes SU(2)$ corresponding to separate rotations of the two spins.

For the case of heavy quarkonia all states fall into spin symmetry quartets which should be degenerate in the non-relativistic limit. In spectroscopic notation ${}^{2S+1}\ell_J$ these quartets consist of the states

$$[n^1\ell_\ell \quad n^3\ell_{\ell-1} \quad n^3\ell_\ell \quad n^3\ell_{\ell+1}]. \tag{5}$$

For the ground states the spin symmetry quartet consists of the η_Q (the 0^- state) and the three polarization directions of the Υ_Q (the 1^- state).

The heavy quarkonia spin symmetry restricts the non-perturbative input to a calculation of processes involving heavy quarkonia. Of particular interest are decays in which the heavy quarks inside the heavy quarkonium annihilate. The annihilation is a short distance process that can be calculated perturbatively in terms of quarks and gluons, while the long distance contribution is encoded in certain matrix elements of quark operators. Logarithmic dependences on the heavy quark mass may be calculated by employing the usual renormalization group machinery.

3 Annihilation Decays of Heavy Quarkonia

The starting point to calculate processes like $\eta_Q \to$ *light hadrons*, $\eta_Q \to \gamma +$ *light hadrons*, or the corresponding decays of the Υ_Q states is the transition operator T for two heavy quarks which annihilate into light degrees of freedom. This will in general be bilinear in the heavy quark fields, such that

$$T(X,\xi) = (-i)\bar{Q}(X+\xi)K(X,\xi)Q(X-\xi)\,, \tag{6}$$

where $K(X,\xi)$ involves only light degrees of freedom and X and ξ correspond to the cms and relative coordinate respectively. If we identify the field Q with the quark and $\bar{Q}$ with the antiquark, so we shall make the large scale m_Q explicit by redefining the fields as

$$Q(x) = \exp(-im_Q(vx))Q_v^{(+)}(x)\,, \quad \bar{Q}(x) = \exp(-im_Q(vx))\bar{Q}_v^{(-)}(x)\,. \tag{7}$$

This corresponds to the usual splitting of the heavy quark momentum into a large part $m_Q v$ and a residual piece k. Inserting this into (6) one finds

$$T = (-i)\exp[-i2m_Q vX]\bar{Q}_v^{(-)}(X+\xi)K(X,\xi)Q_v^{(+)}(X-\xi)\,. \tag{8}$$

The inclusive rate for the decay of a quarkonium $\Psi \to$ *light degrees of freedom* is then given by

$$\Gamma = \langle\Psi| \left[\int d^4X d^4\xi d^4\xi' T(X,\xi)T^\dagger(0,\xi') + \text{h.c.}\right] |\Psi\rangle\,. \tag{9}$$

The next step is to perform an operator product expansion for the non-local product of the quark field operators. This expansion will yield four-quark operators of increasing dimension starting with dim-6 operators. The increasing dimension of these operators will be compensated by inverse powers of the heavy quark mass, so generically the rate takes the form

$$\Gamma = m_Q \sum_{n,i} \left(\frac{1}{m_Q}\right)^{n-2} C(\mathcal{O}_i^{(n)},\mu)\langle\Psi|\mathcal{O}_i^{(n)}|\Psi\rangle|_\mu\,, \tag{10}$$

where $n = 6, 7, \ldots$ is the dimension of the operator and i labels different operators with the same dimension. The coefficients $C(\mathcal{O}_i^{(n)},\mu)$ are related to the short distance annihilation process and hence may be calculated in perturbation theory in terms of quarks and gluons. Once QCD radiative corrections are included, the $C(\mathcal{O}_i^{(n)},\mu)$ acquire a dependence on the renormalization scale μ which is governed by the renormalization group of the effective theory. The rate Γ is independent of μ and hence the μ dependence of $C(\mathcal{O}_i^{(n)},\mu)$ has to be compensated by a corresponding dependence of the matrix elements.

The non-perturbative contributions are encoded in the matrix elements of the local four-quark operators, and the mass dependence of these operators is also expanded in powers of $1/m_Q$ and thus the remaining m_Q dependence of the matrix elements is only due to the states. In terms of the m_Q independent static fields

$$Q_v^{(+)}(x) = h_v^{(+)}(x) + \mathcal{O}(1/m_Q)\,, \quad Q_v^{(-)}(x) = h_v^{(-)}(x) + \mathcal{O}(1/m_Q) \tag{11}$$

one has in total four dim-6 operators

$$A_1^{(C)} = [\bar{h}_v^{(+)}\gamma_5 C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_5 C h_v^{(+)}] \text{ and } A_2^{(C)} = [\bar{h}_v^{(+)}\gamma_\mu^\perp C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_\perp^\mu C h_v^{(+)}]$$

with the color matrix C, where one has the two possibilities $C \otimes C = 1 \otimes 1$ (color singlet) or $C \otimes C = T^a \otimes T^a$ (color octet). These operators do not mix under renormalization, all anomalous dimensions vanish.

There are no dim-7 operators, since these are either proportional to (ivD) and can be rewritten in terms of dim-8 operators by the equations of motion (see (4)) or their forward matrix elements are forbidden by symmetries. At dim-8 one finds 30 local operators

$$\begin{aligned}
B_1^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_5 C h_v^{(-)})]\,[iD_\mu^\perp(\bar{h}_v^{(-)}\gamma_5 C h_v^{(+)})]\,, \\
B_2^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_\mu^\perp C h_v^{(-)})]\,[iD_\perp^\nu(\bar{h}_v^{(-)}\gamma_\nu^\perp C h_v^{(+)})]\,, \\
B_3^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_\perp^\nu C h_v^{(-)})]\,[iD_\mu^\perp(\bar{h}_v^{(-)}\gamma_\nu^\perp C h_v^{(+)})]\,, \\
\\
C_1^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_5 C h_v^{(-)})]\,[\bar{h}_v^{(-)}\gamma_5(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(+)}] + \text{h.c.}\,,
\end{aligned}$$

$$
\begin{aligned}
C_2^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_\mu^\perp C h_v^{(-)}]\,[\bar{h}_v^{(-)}(i\overleftrightarrow{\not{D}}^\perp)C h_v^{(+)}] + \text{h.c.}\,,\\
C_3^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_\perp^\nu C h_v^{(-)})]\,[\bar{h}_v^{(-)}\gamma_\nu^\perp(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(+)}] + \text{h.c.}\,,\\
C_4^{(C)} &= [iD_\perp^\mu(\bar{h}_v^{(+)}\gamma_\perp^\nu C h_v^{(-)})]\,[\bar{h}_v^{(-)}\gamma_\mu^\perp(i\overleftrightarrow{D}_\nu^\perp)C h_v^{(+)}] + \text{h.c.}\,,\\
\\
D_1^{(C)} &= [\bar{h}_v^{(+)}\gamma_5(i\overleftrightarrow{D}_\perp^\mu)C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_5(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(+)}]\,,\\
D_2^{(C)} &= [\bar{h}_v^{(+)}(i\overleftrightarrow{\not{D}}^\perp)C h_v^{(-)}]\,[\bar{h}_v^{(-)}(i\overleftrightarrow{\not{D}}^\perp)C h_v^{(+)}]\,,\\
D_3^{(C)} &= [\bar{h}_v^{(+)}\gamma_\perp^\mu(i\overleftrightarrow{D}_\perp^\nu)C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_\mu^\perp(i\overleftrightarrow{D}_\nu^\perp)C h_v^{(+)}]\,,\\
D_4^{(C)} &= [\bar{h}_v^{(+)}\gamma_\perp^\mu(i\overleftrightarrow{D}_\perp^\nu)C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_\nu^\perp(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(+)}]\,,\\
\\
E_1^{(C)} &= [\bar{h}_v^{(+)}\gamma_5 C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_5(i\overleftrightarrow{D}^\perp)^2 C h_v^{(+)}] + \text{h.c.}\,,\\
E_2^{(C)} &= [\bar{h}_v^{(+)}\gamma_\perp^\mu C h_v^{(-)}]\,[\bar{h}_v^{(-)}(i\overleftrightarrow{\not{D}}^\perp)(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(+)}] + \text{h.c.}\,,\\
E_3^{(C)} &= [\bar{h}_v^{(+)}\gamma_\perp^\mu C h_v^{(-)}]\,[\bar{h}_v^{(-)}(i\overleftrightarrow{D}_\mu^\perp)(i\overleftrightarrow{\not{D}}^\perp)C h_v^{(+)}] + \text{h.c.}\,,\\
E_4^{(C)} &= [\bar{h}_v^{(+)}\gamma_\perp^\mu C h_v^{(-)}]\,[\bar{h}_v^{(-)}\gamma_\mu^\perp(i\overleftrightarrow{D}^\perp)^2 C h_v^{(+)}] + \text{h.c.}\,,
\end{aligned}
$$

where we have defined

$$
\bar{f}(x)(\overleftrightarrow{D})g(x) = \bar{f}(x)(Dg(x)) - (\overline{Df(x)})g(x) \tag{12}
$$

corresponding to (the gauge invariant generalization of) the relative momentum.

Note that we have split these operators into four blocks. The $B_i^{(C)}$ operators contain two derivatives acting on both the heavy quark and the heavy antiquark operators; thus they correspond to a motion of the cms-system of the two heavy quarks relative to the cms-system of the quarkonium. The operators of the $C_i^{(C)}$ type are chosen such that one derivative acts again on both fields, while the other derivative corresponds to the relative motion of the heavy quark-antiquark pair. Finally, the $D_i^{(C)}$ and $E_i^{(C)}$ correspond to two powers of the relative momentum of the two heavy quarks.

4 Renormalization Group Evolution

In (10) a spurious dependence on the renormalization point μ appears, where μ separates long from short distances. The coefficients in front of the matrix elements may be calculated in perturbation theory and are determined at the scale $\mu = m_Q$ from a matching calculation. Once the matching is done one has to evolve down to a small scale which is typical for the matrix elements appearing in (10). As it was discussed in the introduction there are several small scales in a quarkonium and in this context several questions arise. Firstly, if the dimension is not the correct way of counting the relevance of a matrix element, how is the counting of powers done correctly? Secondly, how far should one evolve down using the renormalization group in order to get the "best" result?

For superheavy quarkonia, i.e. the ones one may obviously describe using NRQCD, these problems are easier to discuss. Since the binding is due to instantaneous coulomb gluons, one may write the Lagrangian in Coulomb gauge as

$$\mathcal{L} = \bar{h}_v^{(+)}(ivD)h_v^{(+)} - \bar{h}_v^{(-)}(ivD)h_v^{(-)} + \bar{h}_v^{(+)}\frac{(i\partial^\perp)^2}{2m_Q}h_v^{(+)} + \bar{h}_v^{(-)}\frac{(i\partial^\perp)^2}{2m_Q}h_v^{(-)} \quad (13)$$

since the gluon fields associated with the spatial derivatives $iD^\perp$ are suppressed due to the smallness of the coupling $\alpha_s(m_Q v_{rel}^2)$. Furthermore, coulomb gluons do not appear as dynamical degrees of freedom, rendering the superheavy quarkonium as a real non-relativistic two particle state, with a potential determined from the gluon Greens function $\langle 0|A_0(x)A_0(y)|0\rangle$. In such a case one may perform a Fock state decomposition [4] of the quarkonium of the form

$$|\Psi\rangle = C_0|\bar{Q}Q\rangle + C_1|\bar{Q}Qg\rangle + C_2|\bar{Q}Qgg\rangle + \cdots \quad (14)$$

in which subsequent coefficients are suppressed by powers of v_{rel}, $C_i \sim v_{rel}^i$, since each dynamical gluon is suppressed by a power of v_{rel}. In addition, one has the same selection rules as in Non-Relativistic QED (NRQED), yielding a consistent scheme for an expansion in v_{rel}.

In this way to count powers the matrix elements of the operators $B_i^{(C)}$ and $C_i^{(C)}$ are down by additional powers of v_{rel} compared to their dimension, while the operators of the $D_i^{(C)}$ and $E_i^{(C)}$ type are of order $(m_Q v_{rel})^2$ and thus their matrix elements constitute the first subleading corrections in the power counting scheme along the lines of NRQED. The same argument applies in for the operators in which the two quarks are in a color octet state. Since the quarkonium has to form a color singlet, these contributions have to be of the form of a higher Fock component and are thus suppressed by additional powers of v_{rel}.

However, realistic quarkonia seem not to be close to this limit, and it is well possible, that the v_{rel} counting scheme fails for these systems. In such a case it seems to be useful to count the operators according to their dimension given by the powers of $iD^\perp$. This means that all the matrix elements of the operators $B_i^{(C)}, ..., E_i^{(C)}$ will be taken as the subleading terms. In particular this means that the states of the effective theory are not simply non-relativistic two particle states in which the two heavy quarks are bound by a potential generated by instantaneous, coulomb type gluons. Rather they

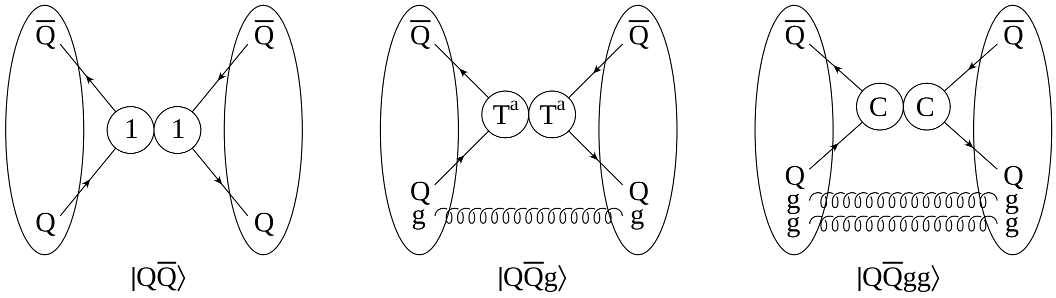

Figure 1: Illustration of the various Fock components. In the standard NRQCD picture the left figure represents the leading contribution.

have to contain dynamical gluons as well such that all the Fock components in (14) are of comparable size. The physical picture is illustrated in fig.1.

The operators $B_i^{(C)}, \ldots, E_i^{(C)}$ are all local operators corresponding to the $1/m_Q$ expansion of the hard kernel. In addition to these local contributions one has also non-local terms originating from single insertions of the Lagrangian of order $1/m_Q^2$ and from double insertions of the Lagrangian of order $1/m_Q$. These contributions correspond to the corrections to the states of the effective theory

Under renormalization the local dim-8 operators do not mix; only the double insertion of the kinetic energy operator of order $1/m_Q$ ($K_1 = K_1^{(+)} + K_1^{(-)}$) mixes into some of the above operators. Denoting this contribution as

$$T_i^{(C)} = \frac{(-i)^2}{2} \int d^4x d^4y \, T[A_i^{(C)} K_1(x) K_1(y)] \tag{15}$$

one obtains in one-loop renormalization group improved perturbation theory two sets of equations for the coefficients of the operators with the spin structure $\gamma_5 \otimes \gamma_5$ [4, 5]

$$\begin{aligned}
C(D_1^{(1)}, \mu) &= C(D_1^{(1)}, m_Q) + \frac{32}{9} \frac{1}{33 - 2n_f} C(T_1^{(8)}, m_Q) \ln \eta \,, \\
C(E_1^{(1)}, \mu) &= C(E_1^{(1)}, m_Q) - \frac{8}{33 - 2n_f} C(T_1^{(1)}, m_Q) \ln \eta \,, \\
C(B_1^{(8)}, \mu) &= C(B_1^{(8)}, m_Q) - \frac{24}{33 - 2n_f} C(T_1^{(8)}, m_Q) \ln \eta \,, \\
C(D_1^{(8)}, \mu) &= C(D_1^{(8)}, m_Q) + \frac{16}{33 - 2n_f} C(T_1^{(1)}, m_Q) \ln \eta \,, \\
&\quad + \frac{20}{3} \frac{1}{33 - 2n_f} C(T_1^{(8)}, m_Q) \ln \eta \,,
\end{aligned} \tag{16}$$

$$C(E_1^{(8)},\mu) = C(E_1^{(8)},m_Q) - \frac{14}{3}\frac{1}{33-2n_f}\,C(T_1^{(1)},m_Q)\ln\eta\,,$$

where n_f is the number of active flavors and $\eta = (\alpha_s(\mu)/\alpha_s(m_Q))$. Furthermore, the coefficients $C(T_i^{(C)},m_Q)$ are the same as the ones for the dim-6 operators $C(A_i^{(C)},m_Q)$ since the kinetic energy operator is not renormalized.

The second set of equations is for the operators with spin structure $\gamma_\mu\otimes\gamma^\mu$ and due to heavy quarkonia spin symmetry one obtains the same equations; all other renormalization group equations are trivial.

The calculation of the coefficients is based on perturbation theory, and hence α_s has to be sufficiently small. Although the final result does not depend on μ, there is still the practical question of how far one should run using the renormalization group. Clearly μ should be chosen to be one of the three, the inverse Bohr radius, the binding energy or Λ_{QCD}. In the worst case Λ_{QCD} is of the order of the inverse Bohr radius, and then the renormalization group evolution has to stop there. In the superheavy case, Λ_{QCD} is small compared to the binding energy and one may run down to the scale set by the binding energy, below this scale one would need to switch to an effective theory involving hadronic instead of QCD degrees of freedom.

5 Hadronic Matrix Elements and Spin Symmetry

A calculation of an annihilation decay then involves to calculate the $C(\mathcal{O}_i^{(n)},\mu)$ at the scale $\mu = m_Q$ by matching the effective theory to full QCD. Once this is done, one may run down to some small scale μ. In the superheavy case μ is of the order of the binding energy of the heavy quarkonium, thereby resumming the well known logarithms of the form $\ln(m_Q/\mu)$ that appear in the calculations of decay rates of heavy p-wave quarkonia. As an example we shall study the decay $\eta_Q \to \gamma$ + *light hadrons* in the next section and consider the first nontrivial corrections to this mode.

The matrix elements of the operators $B_i^{(C)},...,E_i^{(C)}$ as well as the non-local terms are non-perturbative quantities, which are constrained by heavy quarkonia spin symmetry. In order to exploit this symmetry, one may use the usual representation matrices for the spin singlet and spin triplet quarkonia

$$H_1(v) = \sqrt{M}P_+\gamma_5 \text{ for } S=0\,,\quad H_3(v) = \sqrt{M}P_+\epsilon \text{ for } S=1 \qquad (17)$$

where $M \approx 2m_Q$ is the mass of the heavy quarkonium and $P_+ = (1+\not{v})/2$. Using this one finds for the matrix elements of the dim-6 operators

$$\langle\Psi|\bar{h}^{(+)}\Gamma C h^{(-)}\,\bar{h}^{(-)}\Gamma' C h^{(+)}|\Psi\rangle = a^{(C)}(n,\ell)G \;\text{ with }\; G = \mathrm{Tr}(\overline{H}_{2s+1}\Gamma)\mathrm{Tr}(\Gamma' H_{2s+1})$$

Thus for each n and ℓ and for each color combination one finds a single parameter for both the spin singlet and spin triplet quarkonium.

Correspondingly for the dim-8 operators we get

$$\begin{aligned}
\langle\Psi|[iD_\mu^\perp(\bar{h}_v^{(+)}\Gamma C h_v^{(-)})]\,[iD_\nu^\perp(\bar{h}_v^{(-)}\Gamma' C h_v^{(+)})]|\Psi\rangle &= b^{(C)}(n,\ell)(g_{\mu\nu}-v_\mu v_\nu)G \\
\langle\Psi|[\bar{h}_v^{(+)}\Gamma(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(-)}]\,[iD_\nu^\perp(\bar{h}_v^{(-)}\Gamma' C h_v^{(+)})] + \text{h.c.}|\Psi\rangle &= c^{(C)}(n,\ell)(g_{\mu\nu}-v_\mu v_\nu)G \\
\langle\Psi|[\bar{h}_v^{(+)}\Gamma(i\overleftrightarrow{D}_\mu^\perp)C h_v^{(-)}]\,[\bar{h}^{(-)}\Gamma'(i\overleftrightarrow{D}_\nu^\perp)C h_v^{(+)}] &= d^{(C)}(n,\ell)(g_{\mu\nu}-v_\mu v_\nu)G \\
\langle\Psi|[\bar{h}^{(+)}\Gamma(i\overleftrightarrow{D}_\mu^\perp)(i\overleftrightarrow{D}_\nu^\perp)C h_v^{(-)}]\,[\bar{h}^{(-)}\Gamma' C h^{(+)}] + \text{h.c.}|\Psi\rangle &= e^{(C)}(n,\ell)(g_{\mu\nu}-v_\mu v_\nu)G
\end{aligned}$$

For fixed values of n and ℓ one finds that eight parameters are needed to describe the matrix elements of the dim-8 operators.

These matrix elements are non-perturbative, but from vacuum insertion one is lead to assume

$$\begin{aligned}
a^{(1)}(n,0) &= \frac{3}{8\pi}|R_{n0}(0)|^2 \gg a^{(1)}(n,\ell) \text{ for } \ell\neq 0\,, \\
d^{(1)}(n,1) &= \frac{3}{2\pi}|R'_{n1}(0)|^2 \gg d^{(1)}(n,\ell) \text{ for } \ell\neq 1\,, \\
e^{(1)}(n,0) &= \frac{6}{\pi}\,\text{Re}[R''_{n0}(0)R^*_{n0}(0)] \gg e^{(1)}(n,\ell) \text{ for } \ell\neq 0\,,
\end{aligned} \tag{18}$$

$$\begin{aligned}
a^{(1)}(n,0) &\gg a^{(8)}(n,\ell) \text{ for all } n,\ell\,, \\
d^{(1)}(n,1) &\gg d^{(8)}(n,\ell) \text{ for all } n,\ell\,, \\
e^{(1)}(n,0) &\gg e^{(8)}(n,\ell) \text{ for all } n,\ell\,,
\end{aligned} \tag{19}$$

where $R_{nl}(r)$ is the radial wave function of the quarkonium. The same reasoning yields the expectation that $b^{(C)}(n,\ell)$ and $c^{(C)}(n,\ell)$ are small compared to the coefficients that are non-vanishing in vacuum insertion.

In fact, the expectations from vacuum insertion are strengthened by the superheavy case in which vacuum insertion is true to leading order in the v_{rel} expansion. Furthermore, the color octet as well as the operators of the $B_i^{(C)}$ and $C_i^{(C)}$ type are also down by powers of v_{rel}.

6 The Photon Spectrum in $\eta_Q \to \gamma +$ *light hadrons*

The short distance part of the process under consideration is given by the amplitude obtained from the Feynman diagrams depicted in fig.2. The amplitudes are evaluated in QCD and then matched to HQ$\overline{\text{Q}}$ET by performing

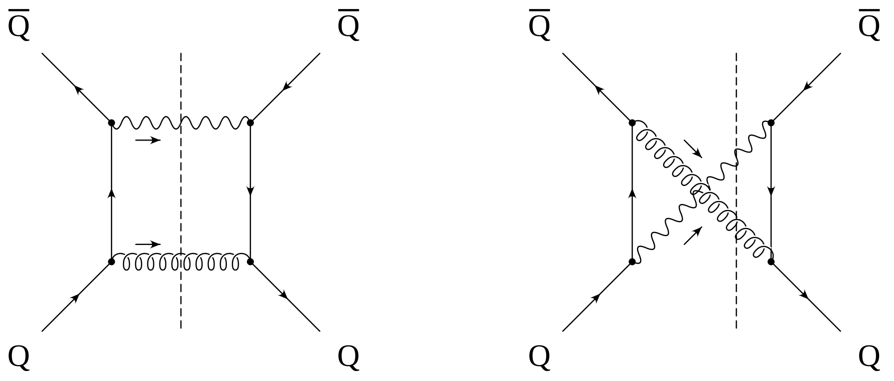

Figure 2: The hard subprocesses for $\eta_Q \to \gamma +$ *light hadrons.*

an expansion in $1/m_Q$. At the matching scale the spectrum, including the first non-trivial corrections, has the form

$$\frac{d\Gamma}{dx} = \mathcal{G}\left[\delta(1-x)\mathcal{A} + \frac{1}{4m_Q^2}\left\{\delta(1-x)\mathcal{B}_0 + \delta'(1-x)\mathcal{B}_1 + \delta''(1-x)\mathcal{B}_2\right\}\right] \tag{20}$$

with

$$\mathcal{G} = \frac{8\pi\alpha_s(m_Q)\alpha_{em}Q_Q^2}{m_Q^2}, \tag{21}$$

where $x := E_\gamma/m_Q$ is the scaled photon energy, $\alpha_{em} = 1/137$ is the electromagnetic coupling and Q_Q is the charge of the heavy quark. At the matching scale $\mu = m_Q$ the coefficients of the δ-functions in (20) are given in terms of the forward matrix elements discussed in the last section and non-local terms originating from the correction in the Lagrangian. One finds from the matching calculation

$$\mathcal{A} = \langle \eta_Q | A_1^{(8)} | \eta_Q \rangle|_{m_Q} \tag{22}$$

$$\begin{aligned} \mathcal{B}_0 = {} & \frac{4}{3}\langle \eta_Q | E_1^{(8)} | \eta_Q \rangle|_{m_Q} - \frac{7}{3}\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_{m_Q} \\ & +(-i)\int d^4x\, \langle \eta_Q | T\{(K_2(x) + \mathcal{L}_{glue}(x))A_1^{(8)}\} | \eta_Q \rangle|_{m_Q} \\ & +\frac{(-i)^2}{2}\int d^4x\, d^4y\, \langle \eta_Q | T\{M_1(x)M_1(y)A_1^{(8)}\} | \eta_Q \rangle|_{m_Q} \end{aligned} \tag{23}$$

$$\mathcal{B}_1 = \frac{1}{2}\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_{m_Q} - \frac{1}{4}\langle \eta_Q | E_1^{(8)} | \eta_Q \rangle|_{m_Q} \tag{24}$$

$$\mathcal{B}_2 = \frac{1}{6}\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_{m_Q} \tag{25}$$

The renormalization group may now be used to run down to some small scale $\mu \sim \Lambda_{QCD}$, thereby extracting logarithms of the form $\log(m_Q/\mu)$ from the matrix elements. Using the one loop result as given in the last section, one has at the small scale

$$
\begin{aligned}
\mathcal{A} &= \langle \eta_Q | A_1^{(8)} | \eta_Q \rangle|_\mu & (26)\\
\mathcal{B}_0 &= \frac{1}{3}\left[4 - \frac{56}{33-2n_f}\ln\eta\right]\langle \eta_Q | E_1^{(8)} | \eta_Q \rangle|_\mu - \frac{1}{3}\left[7 - \frac{288}{33-2n_f}\ln\eta\right]\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_\mu \\
&\quad + \frac{128}{3}\frac{1}{33-2n_f}\ln\eta\langle \eta_Q | D_1^{(1)} | \eta_Q \rangle|_\mu + \frac{1}{33-2n_f}\ln\eta\langle \eta_Q | D_1^{(8)} | \eta_Q \rangle|_\mu \\
&\quad + (-i)\int d^4x\,\langle \eta_Q | T\{(K_2(x) + \mathcal{L}_{glue}(x))A_1^{(8)}\} | \eta_Q \rangle|_\mu & (27)\\
&\quad + \frac{(-i)^2}{2}\int d^4x\,d^4y\,\langle \eta_Q | T\{(M_1(x)M_1(y)A_1^{(8)}\} | \eta_Q \rangle|_\mu \\
\mathcal{B}_1 &= \frac{1}{2}\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_\mu - \frac{1}{4}\langle \eta_Q | E_1^{(8)} | \eta_Q \rangle|_\mu & (28)\\
\mathcal{B}_2 &= \frac{1}{6}\langle \eta_Q | B_1^{(8)} | \eta_Q \rangle|_\mu & (29)
\end{aligned}
$$

Note that the renormalization group flow has induced two operators that have not been present at the matching scale. These type of logarithms has been observed already in the calculation of p-wave quarkonia some time ago [9]; these have been fixed order calculation and the logarithms appear here as infrared singularities. In the effective theory approach they are generated by the renormalization group flow which in addition even resums these terms, since

$$
\begin{aligned}
-\ln\eta &= \ln\left(1 + \alpha_s(m_Q)\frac{\beta_0}{2\pi}\ln\frac{\mu}{m_Q}\right) \\
&= \alpha_s(m_Q)\frac{\beta_0}{2\pi}\ln\frac{\mu}{m_Q} + \frac{1}{2}\left(\alpha_s(m_Q)\frac{\beta_0}{2\pi}\ln\frac{\mu}{m_Q}\right)^2 + \dots, \qquad (30)
\end{aligned}
$$

where $\beta_0 = 11 - 2n_f$.

The expression obtained for the photon spectrum contains δ-function and its derivatives which is obviously unphysical. The origin of this singular behavior is of kinematical nature: At the partonic level, the initial state quarks move with the same velocity and hence act like a single particle, which then decays into two massless objects. Thus this is a two particle decay and so the energies of the final state particles are fixed.

In order to compare with the observed hadron spectrum one has to apply some "smearing" in the sense of [10]; in particular, if one calculates moments of the spectrum (20) one obtains a sensible answer which may be compared to the observed spectrum[1]. In other words the $1/m_Q$ expansion yields an

[1]In fact, the situation is completely the same as in the endpoint of inclusive $B \to X_u \ell\nu$ or $B \to X_s\gamma$ decays [11], where the $1/m_Q$ expansion also only yields the moments of the observed spectrum.

expansion of the spectrum in terms of singular functions

$$\frac{d\Gamma}{dx} = \sum_{n=0}^{\infty} \frac{1}{n!} M_n \delta^{(n)}(1-x)\,, \tag{31}$$

where the coefficients of the expansion are the moments

$$M_n = \int_0^1 dx \frac{d\Gamma}{dx}(1-x)^n\,. \tag{32}$$

The zeroth moment is simply the total rate

$$\Gamma = \mathcal{G}[\mathcal{A} + \frac{1}{4m_Q^2}\mathcal{B}_0]\,, \tag{33}$$

while the first and the second moments are entirely of order $1/m_Q^2$

$$M_1 = \mathcal{G}\mathcal{B}_1\,, \qquad M_2 = \mathcal{G}\mathcal{B}_2\,. \tag{34}$$

As expected, the first and the second moment are entirely proportional to color octet contributions, while the renormalization group flow induces the color singlet operator $D_1^{(1)}$. In factorization as well as in the NRQCD case all these operators are down by at least two powers of v_{rel} and hence this process would be very much suppressed compared to allowed processes such as $J/\Psi \to \gamma +$ *light hadrons*. Hence this process is a nice testing ground for factorization and the power counting scheme of NRQCD.

7 Conclusions

Unlike systems with a single heavy quark quarkonia-like systems are much more difficult to describe. The simplicity of HQET is due to the fact that only a single small scale appears, which is independent of the heavy mass, and thus a static limit may be performed. In a quarkonium several small scales appear which depend in a non-perturbative way on the heavy mass. The superheavy case resembles very much a QED like system which is bound by coulombic forces. In such a system one has the inverse Bohr radius $m_Q v_{rel}$ and the binding energy $m_Q v_{rel}^2$ as small scales, and if $\Lambda_{QCD} \ll m_Q v_{rel}^2$ the binding is in the sense perturbative that it is due to one gluon exchange which takes in coulomb gauge the form of an instantaneous potential.

In this superheavy case one may transfer practically all the knowledge of the description of positronium in NRQED to the QCD case. However, realistic quarkonia do not seem to be too close to this limit, since the charm and the bottom mass are too small to fulfill $\Lambda_{QCD} \ll m_Q v_{rel}^2$. Thus the power counting scheme of NRQED cannot be naively transferred to QCD, and the way how to organize the effective theory calculation becomes obscure.

The safe way in this case is to rely only on the dimension of the operators involved, thereby taking into account a proliferation of unknown parameters, which all would need to be determined from experiment. It is then of some interest to define observables which are sensitive to specific matrix elements and hence may shed some light on what happens in realistic quarkonia.

In the present paper we have studied the decay $\eta_Q \to \gamma$ + *light hadrons*, which would be strongly suppressed compared to $J/\Psi \to \gamma$ + *light hadrons*, if both J/Ψ and η_Q were superheavy. However, this process is not suppressed due to higher powers of $1/m_Q$ appearing in the heavy mass expansion, rather it is suppressed due to the quarkonia states, with which the matrix elements are taken.

It turns out that the moments of the measured photon spectrum in $\eta_Q \to \gamma$ + *light hadrons* is sensitive to matrix elements which would be strongly suppressed for the superheavy case. Unfortunately, there are not yet data on this process, such that a check, whether it is really suppressed as predicted by the counting of powers of v_{rel} has to wait for future experiments.

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