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AUTOMATIC COMPUTATION OF THREE-LOOP TWO-POINT FUNCTIONS IN LARGE MOMENTUM EXPANSION **

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Abstract

We discuss the calculation of two-point three-loop functions with an arbitrary number of massive propagators and one large external momentum. The relevant subdiagrams are generated automatically. The resulting massless two-point integrals and massive tadpoles are transformed on-line to FORM-expressions ready to be used by existing FORM packages which calculate them analytically. As an example we compute the quartic mass corrections to the photon polarization function.

1 Introduction. In general it is not possible to compute analytically a three-loop propagator diagram with non-vanishing external momentum and internal particles with different masses. However, often a hierarchy between the mass scales exists. A typical example is the case when the magnitude of the external momentum, q, is much larger than all the masses appearing in the diagram. For this kinematical situation there exists a systematic procedure, the Large

[1] The complete paper, including figures, is also available via anonymous ftp at ftp://ttpux2.physik.uni-karlsruhe.de/, or via www at
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Momentum Expansion (LME), which works on the diagrammatic level and delivers an expansion in m^2/q^2 for a given dimensionally regularized diagram [1]. Possible physical applications of LME at three-loop level include e.g. the production of $t\bar{t}$ pairs at e^+e^- colliders and the quark mass effects in the decay rates of Z and Higgs bosons .

In this work we discuss a computer-algebra implementation of the LME at three-loop level. Note that the one- and two-loop cases are technically much simpler. The number of various prototype diagrams to be considered is rather small and all necessary general diagrammatic manipulations (except for the final calculation of the very diagrams produced by LME) are quite possible to do by hand as it was demonstrated in [2]. In contrast to the two-loop case an application of LME to even a single three-loop diagram leads to dozens of subdiagrams and corresponding Feynman integrals. This makes the use of computer algebra methods a must. Once the expansion is constructed it is to be followed by usual UV renormalization. As the second step is straightforward we shall only deal with bare Feynman integrals regulated by taking a generic space time dimension $D = 4 - 2\epsilon$.

We start with listing the rules of LME. Given a specific diagram with a large external momentum one gets an asymptotic expansion for the momentum going to infinity by applying the following prescription:

(i) Generate all subdiagrams of the initial graph such that
   (a) they contain both vertices through which the large momentum goes into and out from the initial graph
   (b) they become one-particle-irreducible when the two vertices are identified.

(ii) Taylor-expand the integrands of these subdiagrams in all small masses and external momenta generated by removing lines from the initial diagram.

(iii) In the initial diagram, shrink the subdiagram to a point and insert the result obtained from the expansion in (ii).

(iv) Sum over all terms.

We will call the subgraphs from step (i) *hard subgraphs* or simply *subgraphs*, while the reduced graph, obtained from step (iii), will be called the *co-subgraph.*

For the non-planar three-loop topology all hard subgraphs are shown in Fig. 1. The corresponding co-subgraphs are obtained from the full graph by shrinking the displayed lines to a point. The number of terms generated by LME increases rapidly with the number of loops in the initial diagram. Also, the relation between the expansion momenta in the subgraph and the loop momenta of the co-subgraph becomes non-trivial. Nevertheless, LME provides well-defined rules and therefore it should be possible to automate it.

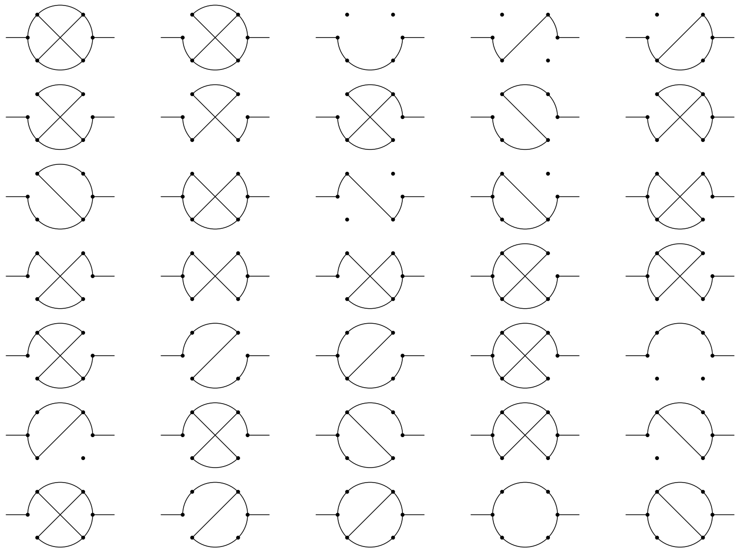

Fig. 1. Possible topologies for the subgraphs of an NO-type diagram.

2 Implementation. We restrict ourselves to three-loop two-point functions containing an arbitrary combination of massive and massless lines. In this section the emphasis will be put on the generation of the terms according to the prescription of Sect. 1. Once they are available, existing program packages written in FORM [3] and based on the method of integration-by-parts [4,5] will be used.

The realization of the LME by a computer can be divided into two steps: (a) generation of the relevant subgraphs and the corresponding co-subgraphs including the determination of their topologies, and (b) distribution of momenta in the subgraph and co-subgraph respecting the relations between them. As for step (a), one only considers topologies, disregarding any properties of lines except their relative positions. Especially one neglects their momentum, mass, particle type, etc. For the representation of a diagram we label its vertices by integers and specify lines by their endpoints. In that way, a topology is described by the collection of its lines.

To generate the subdiagrams we first note that the full graph is always one of the hard subgraphs. The remaining ones are obtained by going through the following steps:

- Remove any combination of lines from the initial diagram.
- Remove the emerging isolated dots and binary vertices.
- Relabel the remaining vertices by $\{1,\ldots,\#\text{vertices}\}$ and build all permutations.
- Compare with a table listing the basic one- and two-loop topologies.

If a subgraph passes the last step, the topology of the corresponding co-subgraph is determined in a similar way. The result of this procedure is a database containing all relevant sub- and co-subgraphs, including information about their topology. Note that no selection criteria bound to line properties have been applied so far. Thus, up to this point the procedure is universal and for each topology this part of the program has to be run only once and for all.

Coming to step (b), we now use the database resulting from (a) and specialize to a specific diagram, including masses and momenta. We bind the direction of the momentum carried by a line to the order of the labels representing this line. Furthermore, for each subgraph the small external momenta are guided through it in a way that they touch as few lines as possible. The final result is the full input for the FORM-packages, together with all the necessary administrative files like makefiles etc.

3 Application. We present the result for the photon polarization function in QED up to three loops and mass corrections up to $\mathcal{O}((m^2/q^2)^2)$. The one- and two-loop results are known since a long time in analytical form [6]. A comparison of these one- and two-loop terms up to $\mathcal{O}((m^2/q^2)^6)$ calculated by our program package with the expansion of the exact results was successful. At three loops the result reads:

$$
\begin{aligned}
\Pi(q^2) = \frac{1}{16\pi^2}\Bigg\{ & \frac{20}{9} - \frac{4}{3}\ln\frac{-q^2}{m^2} + \frac{m^2}{q^2}8 + \left(\frac{m^2}{q^2}\right)^2\left(4 + 8\ln\frac{-q^2}{m^2}\right) \\
& + \frac{\alpha}{\pi}\Bigg[\frac{5}{6} - 4\zeta(3) - \ln\frac{-q^2}{m^2} + \frac{m^2}{q^2}\left(-12\ln\frac{-q^2}{m^2}\right) \\
& + \left(\frac{m^2}{q^2}\right)^2\left(\frac{2}{3} + 16\zeta(3) - 10\ln\frac{-q^2}{m^2} - 12\ln^2\frac{-q^2}{m^2}\right)\Bigg] \\
& + \left(\frac{\alpha}{\pi}\right)^2\Bigg[-\frac{1703}{432} + \zeta(2)\left(8\ln 2 - \frac{23}{3}\right) - \frac{173}{72}\zeta(3) + 10\zeta(5) \\
& + \left(\frac{47}{24} - \frac{4}{3}\zeta(3)\right)\ln\frac{-q^2}{m^2} - \frac{1}{6}\ln^2\frac{-q^2}{m^2} + \frac{m^2}{q^2}\Big(33 + \zeta(2)(48\ln 2 \\
& -46) + \frac{7}{3}\zeta(3) - \frac{70}{3}\zeta(5) + \frac{43}{6}\ln\frac{-q^2}{m^2} + 7\ln^2\frac{-q^2}{m^2}\Big) \\
& + \left(\frac{m^2}{q^2}\right)^2\Big(\frac{1214}{27} + \frac{640}{9}\zeta(3) - 48\zeta(4) - 20\zeta(5) + 8B_4 \\
& + \left(\frac{1003}{36} + \zeta(2)\,(96\ln 2 - 92) - \frac{104}{3}\zeta(3)\right)\ln\frac{-q^2}{m^2} \\
& + \frac{119}{6}\ln^2\frac{-q^2}{m^2} + \frac{32}{3}\ln^3\frac{-q^2}{m^2}\Big)\Bigg]\Bigg\}\ldots
\end{aligned}
\tag{1}
$$

where $B_4 \approx -1.762800$ [5]. The imaginary part of Eq. (1) is in agreement with [7] which is an important check for our programs. Terms up to $\mathcal{O}((m^2/q^2)^6)$ have been calculated by our program package and successfully compared with the results obtained by different methods [8].

To conclude, the manual effort for calculating a three-loop propagator-diagram in the Large Momentum Expansion has been reduced to its minimum. This will find several applications in physics, e.g. the $\mathcal{O}(\alpha_s^2(m^2/q^2)^n)$-terms ($n = 3, 4, \ldots$) of the photon polarization function. The use of these terms as input for the Padé method to get the polarization function in the whole energy range will certainly contribute to a further improvement and stabilization of the results of [8]. From the technical point of view, the program should be extendible also to the hard mass procedure, which will be one of the future projects.

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