# COMPUTING TECHNIQUES FOR TWO-LOOP CORRECTIONS TO ANOMALOUS MAGNETIC MOMENTS OF LEPTONS

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Two-loop electroweak corrections to the electron, muon, and tau lepton anomalous magnetic moments have been computed recently. Effects of hadronic contributions to the photon propagator in two-loop QED corrections have also been reanalysed. The common technique used in both calculations was asymptotic expansion of Feynman diagrams in the ratio of lepton mass to some heavy mass scale. In this talk we present some details of this method paying particular attention to the expansion of multi-scale diagrams.

*Key words:* Standard Model, Feynman Diagram, Anomalous Magnetic Moment
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## 1 Introduction

Anomalous magnetic moments of leptons $a_l \equiv (g_l - 2)/2$ have been providing stimulus for the development of quantum field theory as well as its precise test ever since the first measurement of $a_e$ in 1947 [1]. Later precise measurements of $a_e$ have helped to establish quantum electrodynamics as one of the best tested physical theories. Since the relative contribution of heavy fields to $a_l$ scales like $m_l^2$, muons are more sensitive to hadronic ($a_\mu^{\rm had}$) or electroweak ($a_\mu^{\rm EW}$) effects (and to possible new physics) than electrons. An upcoming experiment E821 [2] at Brookhaven National Laboratory is going to be the first one to test the electroweak loop effects in $a_\mu$. Because of the high precision of

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this experiment a full calculation of two-loop electroweak effects was carried out [3,4]. It was found that the two-loop effects reduce $a_\mu^{\rm EW}$ by 22.6% from $195 \times 10^{-11}$ to $151(4) \times 10^{-11}$.

In the present talk we present some details of those calculations. The method of asymptotic expansions will be discussed and explained with an example of a non-trivial two-loop diagram with three mass scales. The same method can also be applied to the determination of the kernel functions for the hadronic effects in the photon propagator; their application to the calculation of $a_l^{\rm had}$ will be presented.

## 2 Asymptotic Expansions with Multiple Mass Scales

For the two-loop Feynman diagrams with more than one mass scale there exist practically no analytical results. Therefore, a calculation of two-loop electroweak corrections to $a_l$ can only be done using one of two methods: numerical integration or analytical expansion in the small parameters, like the ratio of the lepton and the weak boson masses. The latter approach has several advantages: analytical results can be obtained to any order in the small mass ratio (in particular we can get an exact coefficient of the large logarithms); divergent diagrams are easily treated; and the summation over the large number of diagrams does not induce rounding errors present in the numerical approach. The expansion has to be done carefully even if a diagram contains widely separated mass scales: the integration over virtual momenta and the expansion of the integrand do not in general commute. A rigorous procedure for these so called asymptotic expansions was developed [5]. Below we describe an application of this formalism to problems with more than one expansion parameter, i.e. with three or more different masses or external momenta.

*Example of a 3-scale-problem*

We consider as a practical example a contribution to muon's $g-2$ in which the external photon couples to a closed $\tau$-loop, which in turn is connected to the muon line by a Z boson and a photon (fig. (1); we omit the external photon coupling in the figure.)

We are interested in an expansion in the muon and the $\tau$ masses. In the first step, we consider all masses and momenta as small compared to $M_Z$. Following the rules of the Heavy Mass Expansion (HME) we have to sum over subgraphs $\gamma$ of the initial diagram $\Gamma$ such that each $\gamma$ contains all the lines with the large mass and consists of connectivity components which are 1 PI w.r.t. lines with small masses [5].

This rather formal statement is easy to understand with the help of fig. (1). In each subgraph we expand the propagators in small masses and momenta (in our example $m_\tau$, $m_\mu$ and the external momentum $q$, $q^2 = m_\mu^2$.) A co-subgraph (the remainder of the original diagram in which a subgraph has been contracted to a point) has to be evaluated exactly. For instance, the first subgraph in fig. (1) is the initial two-loop diagram, expanded in $m_\tau$, $m_\mu$ and $q$. In the following two subgraphs one $\tau$ line forms a co-subgraph.

After the first step, the heavy mass scale $M_Z$ has been integrated out. The last co-subgraph, however, still contains two different scales $m_\tau$ and $m_\mu$. In order to deal with it, we now consider $m_\tau$ as the large mass scale and apply the HME to this co-subgraph (second line in fig. (1)). Iterative application of the HME along these lines allows a consistent expansion of a given diagram in terms of powers and logarithms of the masses and momenta, as long as there is a clear hierarchy between the scales. We end up with one- or two-loop single scale integrals which can be evaluated analytically.

In the calculation of $a_l$ we encounter diagrams with both $W$ and $Z$ bosons; their masses are similar. In such case we may proceed in the following way: first expand in the difference of the two similar scales. The number of scales is now reduced by one, and we may continue with the HME.

## 3 Parametrization of two-loop electroweak contributions to $g-2$ of the electron, muon and $\tau$ lepton

Including two-loop corrections, the electroweak contribution to the anomalous magnetic moment of a lepton with mass $m_l$ can be parametrized as a product of the one-loop expression and a correction factor $\left(1 + C\frac{\alpha}{\pi}\right)$

$$a_l^{\mathrm{EW}} = a_l^{1-\mathrm{loop}}\left(1 + C\frac{\alpha}{\pi}\right), \qquad a_l^{1-\mathrm{loop}} \approx \frac{5}{3}\frac{G_\mu m_l^2}{8\sqrt{2}\pi^2} \tag{1}$$

It is natural to separate the subset of the two loop electroweak contributions which contain a closed fermion loop:

$$C = C^{\mathrm{ferm}} + C^{\mathrm{bos}}\,. \tag{2}$$

A calculation of the two-loop electroweak contributions to $a_\mu$ was presented in some detail in [3,4,6]. The results for $a_e$ and $a_\tau$ were presented in ref. [4].

Below we discuss in some detail the connection between $a_{e,\tau}$ and $a_\mu$.

*Bosonic Contributions*

In ref. [4] it was found that due to accidental cancellations the bosonic contribution to $a_\mu$ is very well approximated by terms containing large logarithms:

$$C_\mu^{\rm bos}(2-{\rm loop}) \approx \ln\frac{M_W^2}{m_\mu^2}\cdot\left(-\frac{13}{3}+\frac{92}{15}s_W^2-\frac{184}{15}s_W^4\right)\,. \tag{3}$$

In diagrams without fermion loops the only mass scales are the lepton mass and much larger masses of the weak bosons. Therefore, the magnetic anomalies of $e$ and $\tau$ leptons can be obtained from eq.(3) by replacing $m_\mu$ by $m_e$ or $m_\tau$. The numerical values are presented in table (1) [2].

*Fermionic Contributions*

In order to obtain the bosonic contributions to $g-2$ of the electron and the $\tau$ lepton a simple rescaling of masses in the muon result could be performed. This cannot be done so easily in the diagrams containing a fermion loop connected to the muon line by a photon and a $Z$ boson because the internal fermion introduces an additional mass scale.

In the case of the $\tau$ we put $m_\tau = m_c$ and employ the formula for $\Delta C^\mu_{1d}$ in eq. (16) of ref. [3] for the charm quark. The final result for the fermionic contributions to $a_\tau$ is

$$\begin{aligned} C_\tau^{\rm ferm} = & -\frac{9}{5}\left(\ln\frac{M_Z^2}{m_\tau^2}+\frac{1}{3}\ln\frac{M_Z^2}{m_b^2}\right)-\frac{109}{30}-\frac{8}{45}\pi^2-\frac{3}{16}\frac{m_t^2}{s_W^2M_W^2} \\ & -\frac{3}{10s_W^2}\ln\frac{m_t^2}{M_W^2}-\frac{8}{5}\ln\frac{m_t^2}{M_Z^2}-\frac{7}{10s_W^2}+\Delta C_{Higgs} \end{aligned} \tag{4}$$

For the electron the fermionic contribution is

[2] For the entries in the table the full formula including the non-logarithmic terms was used.

$$C^{\rm ferm}_{\rm electron} = -\frac{18}{5}\ln\frac{(m_u m_c M_Z)^{4/3}}{(m_d m_s m_b)^{1/3} m_e m_\mu m_\tau} - \frac{31}{10} + \frac{8}{15}\pi^2 - \frac{7}{10 s_W^2}$$
$$-\frac{3}{16}\frac{m_t^2}{s_W^2 M_W^2} - \frac{3}{10 s_W^2}\ln\frac{m_t^2}{M_W^2} - \frac{8}{5}\ln\frac{m_t^2}{M_Z^2} + \Delta C_{Higgs} \qquad (5)$$

## 4 Higher Order Hadronic Contributions

The main goal of the forthcoming BNL experiment [2] is to measure electroweak loop effects; at the same time, however, it will also be sensitive to higher order hadronic effects. These are $\mathcal{O}(\alpha^3)$ and fall into two different classes: diagrams with hadronic self energy insertion in the photon propagator, and light-by-light scattering diagrams. The latter will not be discussed here. For the former, the high precision of the BNL experiment requires a reconsideration of the corresponding kernel functions [7]. To this end, it is straightforward to apply the asymptotic expansion method, as presented in the first section, also to the analytical calculation of the higher order kernel functions for the muon and the electron. The idea is to write the original three-loop diagrams with hadronic self energy insertion in the photon propagator as a dispersion integral over an effective photon "mass" $\sqrt{s}$. Since the lower limit of the dispersion integral is the $\pi^+\pi^-$ threshold, we may perform an expansion of the corresponding Feynman diagram w.r.t. $\frac{m^2}{s}$, where $m$ is the muon or the electron mass. In this way the problem is reduced to computing single scale diagrams; the final integration over $s$ has to be done numerically, including experimental data for $R(s) = \frac{\sigma(e^+e^-\to hadrons)}{\sigma(e^+e^-\to\mu^+\mu^-)}$. Main advantages of this approach are that it is very simple to obtain numerically well-behaved analytical expressions with sufficient accuracy and the error in the numerical integration is reduced to the final $s$-integral over data. Along these lines the higher order hadronic contributions to $g-2$ of leptons have been computed and previous calculations [8] could be checked by an independent method [7]. For the muon a shift by $-11\cdot 10^{-11}$ w.r.t. previous work [9] was found – which is somewhat smaller but still of the same order of magnitude as the experimental precision of the BNL experiment. The results are summarized in table (2).

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## Tables

| Lepton | $C^{\rm bos}(2-\rm loop)$ | $C^{\rm ferm}(2-\rm loop)$ | $a^{\rm EW}_{\rm lepton}$ | $a^{\rm EW}_{2-\rm loop}/a^{\rm EW}_{1-\rm loop}$ |
|---|---|---|---|---|
| $\mu$ | $-47.3$ | $-50.0$ | $151(4)\cdot 10^{-11}$ | $-22.6\%$ |
| $e$ | $-85.4$ | $-64.1$ | $3.0(1)\cdot 10^{-14}$ | $-35\%$ |
| $\tau$ | $-27.1$ | $-37.8$ | $4.7(1)\cdot 10^{-7}$ | $-15\%$ |

Table 1: Two-loop electroweak contributions to the anomalous magnetic moment of the muon, electron and $\tau$ lepton, separated into the bosonic and fermionic subsets, $C_{\rm bos}$ and $C_{\rm ferm}$, and the total electroweak correction $a_l = (g_l-2)/2$. Here, $\sin^2\theta_W = 0.224$ and $M_H = 250$ GeV were used; in $C^{\rm ferm}$ terms $\mathcal{O}(1-4\sin^2\theta)$ were neglected. In $C^{\rm bos}$ $\sin^2\theta$ was used as an expansion parameter and the first four terms of the expansions were retained. The two-loop corrections reduce the one-loop results by the amount given in the last column.

| Lepton | $a_{\text{had.,leading}}$ (ref. [10]) | $a_{\text{had.,higher order}}$ |
|---|---|---|
| $\mu$ | $7023.5 \pm 58.5 \pm 140.9 \cdot 10^{-11}$ | $-101 \pm 6 \cdot 10^{-11}$ |
| $e$ | $1.8847 \pm 0.0165 \pm 0.0375 \cdot 10^{-12}$ | $-2.25 \pm 0.05 \cdot 10^{-13}$ |
| $\tau$ | $338.30 \pm 1.97 \pm 9.12 \cdot 10^{-8}$ | $7.6 \pm 0.2 \cdot 10^{-8}$ |

Table 2: Higher order hadronic contributions to the $g-2$ of leptons.

**Figures**

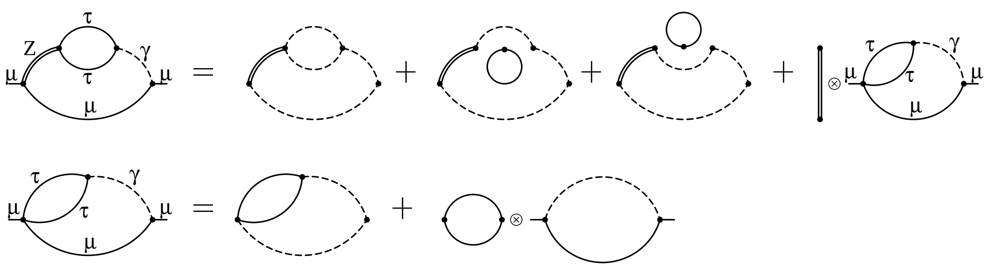


Figure 1: Example of the decomposition of a three-scale-problem into single scale diagrams. Dashed lines denote massless propagators.