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Eight–component differential equation for leptonium

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Abstract

It is shown that the potential for lepton-antilepton bound states (leptonium) is the Fourier transform of the first Born approximation to the QED scattering amplitude in an 8-component equation, while 16-component equations are excluded. The Fourier transform is exact at all *cms* energies $-\infty < E < \infty$; the resulting atomic spectrum is explicitly CPT-invariant.

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I. INTRODUCTION

In muonium $e^-\mu^+$, the mass m_2 of the heavier particle is so large that the kinetic energy $p_2^2/2m_2$ can be added as a recoil correction to the electron's Dirac Hamiltonian. The resulting Dirac equation has 8 components, with the Pauli matrices $\boldsymbol{\sigma}_2$ in the hyperfine operator [1,2]. However, a similar 8–component equation exists which avoids the expansion in terms of $1/m_2$ [3,4]. Despite its asymmetric form, it reproduces the energy levels also for positronium e^-e^+ to order α^4 [5]. Unfortunately, its complicated derivation via Breit operators introduced a low-energy approximation, which is removed in the present paper. The new interaction is simply the Fourier transform of the QED Born term in the *cms*, valid at all energies, $-\infty < E < \infty$ (by crossing symmetry, $E < 0$ for $e^-\mu^+$-scattering refers to $E > 0$ for $e^+\mu^-$-scattering). The equation is written in the *cms*, $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$, in units $\hbar = c = 1$:

$$\frac{1}{2}E^2\psi = (K_0 + K_I)\psi\,, \quad K_0 = E\mathbf{p}\boldsymbol{\alpha} + \frac{1}{2}(m_2 + \beta m_1)^2\,, \tag{1}$$

$$\boldsymbol{\alpha} = \gamma_5\boldsymbol{\sigma}_1\,, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2}$$

The Pauli matrices $\boldsymbol{\sigma}_2$ appear only in the interaction operator K_I. The proof that the equation with $K_I = 0$ describes two free Dirac particles instead of one is given in section II. The construction of K_I from the Born approximation is done in section III. Orthogonality relations are derived in section IV.

Before going into details, we remind the reader that in a 16-component version,

$$E\psi = (H_1 + H_2 + H_{12})\psi\,, \quad H_i = \mathbf{p}_i\boldsymbol{\alpha}_i + m_i\beta_i\,, \tag{3}$$

the energy levels to order α^4 are not reproduced when H_{12} is the Fourier transform of the Born approximation, $H_{12} = V(1 - \boldsymbol{\alpha}_1\boldsymbol{\alpha}_2)$ $(V = -\alpha/r\,,\ \alpha = e^2)$. This interaction was improved by Breit [6] and was later amended by energy-transfer [7,8] and by positive-energy projectors [9]. The method is now quite successful in atomic theory, where the nucleus produces an external potential for the electrons [8]. In the previous derivation of (1) via Breit operators [3,4], a simpler prescription was used which sets the (large) squares of Breit operators equal to zero. However, the connection with perturbative QED was then incomplete. In the present derivation, the connection with the Born series is straightforward. With loop graphs included, all α^6-effects in binding energies should appear, for arbitrary values of m_1/m_2.

For $V = 0$, (1) and (3) are equivalent. In the first Born approximation, K_I of (1) and H_{12} of (3) are linear in V. When (3) is reduced to 8 components, quadratic terms appear which contradict the leptonium energy levels to order α^4. Some of these terms were eliminated by a canonical transformation, the rest by projectors.

To understand the significance of the 8-component theory, one may note that 8 is the minimum number of components for a Lorentz- and parity invariant theory. An irreducible representation of the Lorentz group for a single lepton requires only two components, for example the right-handed ψ_R in a chiral basis. It is the parity transformation, $\psi'(-\mathbf{r}) = \beta\psi(\mathbf{r})$, which exchanges ψ_R with the left-handed components ψ_L (in the chiral or Weyl basis, γ_5 and β of (2) exchange their roles). For two leptons, the necessary $2 \times 2 = 4$ components are doubled by the matrix β of (2), which corresponds to $\beta_1\beta_2$ in a 16-component formalism.

The separate matrices β_1 and β_2 of the 16-component formalism provide another unnecessary doubling.

We end this section by casting (1) into a more familiar form. We divide it by E and define the reduced mass μ and reduced energy ϵ:

$$\mu = m_1 m_2/E\,, \quad \epsilon = (E^2 - m_1^2 - m_2^2)/2E\,, \tag{4}$$

$$\epsilon\psi = (\mathbf{p}\boldsymbol{\alpha} + \mu\beta + K_I/E)\psi\,. \tag{5}$$

We also anticipate the result (41) for K_I, which gives:

$$K_I/E = V - i\gamma_5 V\mathbf{p}(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)/E\,. \tag{6}$$

The difference to an effective Dirac equation is obviously restricted to the hyperfine operator $-iV\mathbf{p}(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2)/E$. As the *cms* energy E is close to $m_1 + m_2$ for ordinary bound states, the replacement of the familiar $1/m_2$ by $1/E$ is a recoil correction. The essential new feature is that the operator $V\mathbf{p}$ is not hermitian. An 8-component equation with a hermitian operator does not reproduce the positronium bound states.

Anomalous magnetic moments are neglected for the time being, so that the present formulation does not cover atomic hydrogen. The annihilation graph in the $^3S_1$ state of positronium is also neglected. The hyperfine operator will be further simplified in appendix A. Its connection with the previous approximate operator [4,5] is given in appendix B. Appendix C contains a spin summation, which is included for demonstration of covariance.

II. EIGHT-COMPONENT SPINORS FOR TWO LEPTONS

Let u_{ig} and u_{if} denote the large and small components of the Dirac spinor u of a particle i in an orbital:

$$\mathbf{p}_i\boldsymbol{\sigma}_i\, u_{ig} = (\pi_i^0 + m_i)\, u_{if}\,, \quad \mathbf{p}_i\boldsymbol{\sigma}_i\, u_{if} = (\pi_i^0 - m_i)\, u_{ig}\,, \quad \pi_i^0 = E_i - V_e(\mathbf{r}_i)\,. \tag{7}$$

Here V_e is a possible external potential, as in He-atoms. Leptonium has $V_e = 0$, of course. The chiral operator γ_{5i} exchanges u_{ig} with u_{if}. Out of the 4 products $u_{1g}u_{2g}\ldots u_{1f}u_{2f}$, we keep in this section only the two combinations of total chirality $+1$ (the eigenvalue of $\gamma_{51}\gamma_{52}$):

$$\begin{aligned} \frac{1}{\sqrt{2}}(u_{1g}u_{2g} + u_{1f}u_{2f}) &\equiv u_{dg}\,, \\ \frac{1}{\sqrt{2}}(u_{1f}u_{2g} + u_{1g}u_{2f}) &\equiv u_{df}\,. \end{aligned} \tag{8}$$

Inspection shows that these combinations satisfy the following equations:

$$\begin{aligned} (m_2\mathbf{p}_1\boldsymbol{\sigma}_1 + m_1\mathbf{p}_2\boldsymbol{\sigma}_2)\, u_{dg} &= (m_2\pi_1^0 + m_1\pi_2^0)\, u_{df}\,, \\ (m_2\mathbf{p}_1\boldsymbol{\sigma}_1 - m_1\mathbf{p}_2\boldsymbol{\sigma}_2)\, u_{df} &= (m_2\pi_1^0 - m_1\pi_2^0)\, u_{dg}\,. \end{aligned} \tag{9}$$

Apart from the m_1-terms, these equations have the same structure as a single-particle Dirac equation. We therefore introduce an 8×8-matrix β (which has the same eigenvalues as

$\beta_1\beta_2$) and an anticommuting matrix γ_5, $\gamma_5\beta = -\beta\gamma_5$, and combine u_{dg} and u_{df} into one 8-component double spinor u_d:

$$u_d = \begin{pmatrix} u_{dg} \\ u_{df} \end{pmatrix}, \quad \gamma_5(m_2\mathbf{p}_1\boldsymbol{\sigma}_1 + \beta m_1\mathbf{p}_2\boldsymbol{\sigma}_2)\,u_d = (m_2\pi_1^0 - \beta m_1\pi_2^0)\,u_d\,. \tag{10}$$

γ_5 and β have already been given in (2), understanding that the number 1 is a 4×4-matrix in spin space. For $m_1 = m_2$, (10) should also describe two non-interacting electrons in a helium atom. Their Born scattering amplitude for $V_e \neq 0$ will be constructed in (35) below, but our construction of the interaction operator in the differential equation neglects V_e. Turning now to the free two-body problem, we set $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$,

$$\pi_1^0 = E_1 = \frac{1}{2E}(E^2 + m_1^2 - m_2^2)\,, \quad \pi_2^0 = E_2 = \frac{1}{2E}(E^2 - m_1^2 + m_2^2)\,. \tag{11}$$

A factor $m_2 - \beta m_1$ can then be divided off from the right-hand side of (10), the result being:

$$(\boldsymbol{\alpha}^c\mathbf{p} + \mu\beta - \epsilon)\,u_d = 0\,, \tag{12}$$

$$\boldsymbol{\alpha}^c = \gamma_5\boldsymbol{\sigma}_1^c = \gamma_5(m_2\boldsymbol{\sigma}_1 - \beta m_1\boldsymbol{\sigma}_2)/(m_2 + \beta m_1)\,. \tag{13}$$

Including the space-dependence, the free double-spinor wave function in the *cms* is:

$$\psi_d(\mathbf{r}) = u_d\,e^{i\mathbf{k}_1\mathbf{r}_1}e^{i\mathbf{k}_2\mathbf{r}_2} = u_d\,e^{i\mathbf{k}\mathbf{r}}\,, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2\,. \tag{14}$$

It differs from the corresponding free-particle solution of (1), which has $\boldsymbol{\alpha}^c$ replaced by $\boldsymbol{\alpha}$. This difference may be one reason why (1) has not been discovered 60 years ago. During these 60 years, many different formalisms have been developed. Bethe and Salpeter advocated the use of four-dimensional integral equations (with a relative time as fourth integration variable), again with 16 components. Although the relative-time concept turned out to be useless, one learned to find bound states from integral equations in momentum space, now in three dimensions [2]. This method avoids the Fourier transformation. Having elaborated these momentum space methods, one may be unwilling to return to differential equations, particularly if these require such strange matrices as $\boldsymbol{\sigma}_1^c$. On the other hand, the advantage of the present formulation survives also in 8-component integral equations in momentum space. It could become essential also in quarkonium models.

To establish the connection between (1) and (12), we first define two mass operators:

$$m_\pm = m_2 \pm \beta m_1\,, \quad m_+m_- = m_2^2 - m_1^2\,, \quad m_\pm^2 = m_1^2 + m_2^2 \pm 2m_1m_2\beta\,. \tag{15}$$

The expression for m_+^2 has already been used in (5). From $\gamma_5\beta = -\beta\gamma_5$, one finds:

$$\gamma_5 m_+ = m_-\gamma_5\,. \tag{16}$$

The Dirac spin operators are $\gamma_5\boldsymbol{\sigma}_1$ and $\gamma_5\boldsymbol{\sigma}_2$. With the algebra (15), (16), one easily verifies:

$$(\boldsymbol{\alpha}^c)^2 = (\gamma_5\boldsymbol{\sigma}_1^c)^2 = (m_2\boldsymbol{\sigma}_1 + \beta m_1\boldsymbol{\sigma}_2)(m_2\boldsymbol{\sigma}_1 - \beta m_1\boldsymbol{\sigma}_2)/m_+m_- = 3\,. \tag{17}$$

Comparing this with $\boldsymbol{\alpha}^2 = 3$, one sees that there should exist a transformation from $\boldsymbol{\alpha}^c$ to $\boldsymbol{\alpha}$. It was first found in an explicit decomposition of u_d into u_{dg} and u_{df}, and of the spin states χ_{12} into χ_s (singlet) and χ_t (triplet, eq. (5.14) in [3]). Its compact Dirac form is:

$$\psi_d = c\psi\,, \quad c = (m_+ m_-)^{-1/2}\left[m_+ - 2m_1\beta P_s\right]\,. \tag{18}$$

P_s is the projector on the singlet spin state χ_s. In the following, the corresponding triplet projectors on the three states χ_t will also be needed:

$$P_s = \frac{1}{4}(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\,, \quad P_t = \frac{1}{4}(3 + \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\,. \tag{19}$$

We also define combinations of Pauli matrices,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2\,, \quad \Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\,, \quad \boldsymbol{\sigma}^\times = \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2\,, \tag{20}$$

which have the following products with P_s and P_t:

$$\boldsymbol{\sigma} P_s = P_s \boldsymbol{\sigma} = 0\,, \quad \Delta\boldsymbol{\sigma} P_t = P_s \Delta\boldsymbol{\sigma}\,, \quad \Delta\boldsymbol{\sigma} P_s = P_t \Delta\boldsymbol{\sigma}\,, \quad \boldsymbol{\sigma}^\times P_s = P_t \boldsymbol{\sigma}^\times\,. \tag{21}$$

c is not unitary, its inverse being:

$$c^{-1} = (m_+ m_-)^{-1/2}(m_+ - 2m_1\beta P_t)\,, \tag{22}$$

which is checked by using $P_s + P_t = 1$, $P_s P_t = 0$. An important property of c is:

$$c^{-1}\gamma_5 = \gamma_5 c\,. \tag{23}$$

By parity invariance, the operators (20) are always accompanied by one factor γ_5, which in view of (23) replaces c^{-1} by c:

$$c\,\boldsymbol{\sigma}\,c = \boldsymbol{\sigma}\,m_+/m_-\,, \quad c\,\Delta\boldsymbol{\sigma}\,c = \Delta\boldsymbol{\sigma}\,, \quad c\,\boldsymbol{\sigma}^\times\,c = \boldsymbol{\sigma}^\times\,. \tag{24}$$

Expressing $\boldsymbol{\sigma}_1$ as $\frac{1}{2}(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})$, one finds for the combination (13):

$$c\,\boldsymbol{\sigma}_1^c\,c = \boldsymbol{\sigma}_1\,. \tag{25}$$

Thus the transformation (18) leads from (12) to the free equation (1). Other forms of that equation are generated by additional transformations d satisfying:

$$\beta d = d\beta \quad d^{-1}\gamma_5 = \gamma_5 d\,. \tag{26}$$

III. THE BORN APPROXIMATION AND ITS FOURIER TRANSFORM

The Lorentz-invariant T-matrix for lepton-antilepton scattering from initial orbitals 1,2 into final orbitals $1', 2'$ has the following Born approximation:

$$T/4\pi = \alpha \bar{u}_1' \gamma^\mu u_1 \bar{u}_2' \gamma_\mu u_2 / t = \alpha u_1'^\dagger u_2'^\dagger (1 - \boldsymbol{\alpha}_1\boldsymbol{\alpha}_2) u_1 u_2 / t\,, \tag{27}$$

with $\boldsymbol{\gamma}_i = \beta_i \boldsymbol{\alpha}_i$, and $t = q^\mu q_\mu = q_0^2 - \mathbf{q}^2$ being the square of the 4-momentum transfer. In the *cms*, the arguments of the free Dirac spinors are $\mathbf{k}$ and $-\mathbf{k}$ in the initial state and $\mathbf{k}'$ and $-\mathbf{k}'$ in the final state, and $\mathbf{q} = \mathbf{k} - \mathbf{k}'$, $q_0 = 0$. Remembering

$$u_1'^\dagger u_1 u_2'^\dagger u_2 = \left(u_{1g}'^\dagger u_{1g} + u_{1f}'^\dagger u_{1f}\right)\left(u_{2g}'^\dagger u_{2g} + u_{2f}'^\dagger u_{2f}\right) \tag{28}$$

etc., one sees that T cannot be written as a bilinear in u_d and $u_d'^\dagger$. One also needs the states of total chirality -1, which will be called w_{dg} and w_{df}:

$$w_{dg} = \frac{1}{\sqrt{2}}(u_{1g}u_{2g} - u_{1f}u_{2f}),$$

$$w_{df} = \frac{1}{\sqrt{2}}(u_{1f}u_{2g} - u_{1g}u_{2f}), \tag{29}$$

$$T/4\pi = \alpha\left[u_d'^\dagger(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)u_d + w_d'^\dagger(1+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)w_d\right]/t\,. \tag{30}$$

This form still has its 16-component character, as the replacement of u_1u_2 by u_d and w_d is just a unitary transformation. However, in addition to the separate equations for u_d and w_d (the equation for w_d has $\boldsymbol{\sigma}_2$ replaced by $-\boldsymbol{\sigma}_2$) there exist also coupled equations, with $\pi^0 = \pi_1^0 + \pi_2^0$ and $p_\pm = \mathbf{p}_1\boldsymbol{\sigma}_1 \pm \mathbf{p}_2\boldsymbol{\sigma}_2$:

$$w_d = m_+^{-1}(\pi^0 - \gamma_5 p_+)u_d\,, \quad u_d = m_+^{-1}(\pi^0 - \gamma_5 p_-)w_d\,, \tag{31}$$

which can be verified explicitly from (8) and (29). By means of (31), T can be written in terms of a single 8×8-matrix M,

$$T/4\pi = \alpha\, w_d'^\dagger\, M\, u_d\,/t\,, \tag{32}$$

$$M = (\pi^0 - \gamma_5 p_-')m_+^{-1}(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) + (1+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)m_+^{-1}(\pi^0 - \gamma_5 p_+)\,. \tag{33}$$

The operators proportional to π^0 combine into $2\pi^0$, while the operators containing p_-' combine as follows:

$$p_-'\,(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) = \mathbf{p}(\Delta\boldsymbol{\sigma} + i\boldsymbol{\sigma}^\times) = p_-(1 - \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\,. \tag{34}$$

The total momentum $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$ commutes with V and vanishes in the *cms*. There, M reduces to:

$$M = m_+^{-1}\left[2E - \gamma_5(1+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\mathbf{k}\Delta\boldsymbol{\sigma}\right] = 2m_+^{-1}(E - i\gamma_5\mathbf{k}\boldsymbol{\sigma}^\times)\,. \tag{35}$$

An equivalent form of T follows from the elimination of $w_d'^\dagger$ and u_d in (30),

$$T/4\pi = \alpha\, u_d'^\dagger\, M^\dagger\, w_d/t\,, \tag{36}$$

$$M^\dagger = \left[2E - \gamma_5\mathbf{k}'\Delta\boldsymbol{\sigma}(1+\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\right]m_+^{-1} = 2(E + i\gamma_5\mathbf{k}'\boldsymbol{\sigma}^\times)m_+^{-1}\,. \tag{37}$$

This suggests the definition of a second double-spinor wave-function as follows:

$$\chi_d(\mathbf{r}) = e^{i\mathbf{kr}}\, m_+\, w_d\,. \tag{38}$$

When T is expressed in terms of m_+w_d and $w_d'^\dagger m_+$, the factor m_+^{-1} vanishes both in (35) and (37). The interaction in coordinate space follows as:

$$-4\pi\,\alpha/\mathbf{q}^2 = \int d^3r e^{-i\mathbf{k}'\mathbf{r}}\, V\, e^{i\mathbf{kr}}\,, \quad -4\pi\,\alpha\,\mathbf{k}/\mathbf{q}^2 = \int d^3r e^{-i\mathbf{k}'\mathbf{r}}\, V\mathbf{p}\, e^{i\mathbf{kr}}\,, \tag{39}$$

$$-4\pi\,\alpha\,\mathbf{k}'/\mathbf{q}^2 = \int d^3r e^{-i\mathbf{k}'\mathbf{r}}\,\mathbf{p}V\, e^{i\mathbf{kr}}\,. \tag{40}$$

It produces the operator

$$K_I = V(E - i\mathbf{p}\boldsymbol{\sigma}^\times\gamma_5)\,, \quad K_I^\dagger = (E + i\mathbf{p}\boldsymbol{\sigma}^\times\gamma_5)V\,, \tag{41}$$

to be used in (1) and in the corresponding equation for the wave-function $\chi(\mathbf{r}) = e^{i\mathbf{kr}}w$:

$$\frac{1}{2}E^2\chi = (K_0 + K_I^\dagger)\,\chi\,. \tag{42}$$

Although K_I is not hermitian, K_I and $K_I^\dagger$ give equivalent equations, such that the bound state energies may be real.

IV. ORTHOGONALITY RELATIONS AND CONCLUDING REMARKS

When E is replaced by $m = m_1 + m_2$ in the hyperfine operator, (5) and the corresponding equation (42) assume Hamiltonian forms:

$$\epsilon\psi = H\psi\,, \quad \epsilon\chi = H^\dagger\chi\,. \tag{43}$$

Taking the hermitian adjoint of the second equation at reduced energy ϵ' and integrating over $\mathbf{r}$, one obtains

$$(\epsilon - \epsilon')\int \chi'^\dagger\psi = \int \chi'^\dagger(H - H)\psi = 0\,. \tag{44}$$

Thus the non-hermiticity of H is harmless. But in the exact expression (5), the hyperfine operator will remain in the orthogonality relations.

One may also cast (1) into a strictly Hamiltonian form by introducing a secondary 8-component spinor ψ_s:

$$(E - 2\boldsymbol{\alpha}\mathbf{p})\psi = \psi_s\,, \;\; E\psi_s = (m_+^2 + 2K_I)\psi\,. \tag{45}$$

This method is known from the relativistic treatment of spinless particles (for example from the Klein-Gordon equation).

For $V = -\alpha/r$, orthogonality relations are most elegantly derived in a dimensionless scaled variable,

$$\tilde{r} = Er\,, \;\; \partial/\partial\tilde{r} = E^{-1}\partial/\partial r\,, \;\; \tilde{\mathbf{p}} = \mathbf{p}/E\,. \tag{46}$$

Dividing equation (1) by E^2 and setting $E^2 = s$ for convenience, one obtains:

$$\left[\tilde{\mathbf{p}}\boldsymbol{\alpha} + \frac{1}{2}m_+^2/s - \frac{1}{2} + V(\tilde{r})\,(1 - i\tilde{\mathbf{p}}(\boldsymbol{\alpha}\times\boldsymbol{\sigma}_2)\,)\right]\psi(\tilde{\mathbf{r}}) = 0\,. \tag{47}$$

Using the corresponding equation for $\chi^\dagger$, one arrives at:

$$(s_i^{-1} - s_j^{-1})\int \chi_i^\dagger\, m_+^2\, \psi_j d^3\tilde{r} = 0\,, \quad s_i = E_i^2\,, \tag{48}$$

$$\int \chi_i^\dagger\, m_+^2\, \psi_j d^3\tilde{r} = \delta_{ij}\,. \tag{49}$$

Remembering $m_+^2 = m_1^2 + m_2^2 + 2m_1m_2\beta$, this is a simple generalization of the static limit $m_1/m_2 = 0$. For positronium, the small components do not contribute to (49) (a previously proposed substitution $r = E\rho$ [5] gives more complicated orthogonality relations).

Equation (47) is explicitly CPT-invariant: Every bound state s_i has two different eigenvalues E_i, namely $E_i = \sqrt{s_i} \equiv m_{Ai}$ and $E_i = -\sqrt{s_i} \equiv -m_{Ai}$, where m_{Ai} denotes the atomic mass in the state i (an excited atom is heavier than its ground state). The later value belongs to the antiatom of mass $m_{\bar{A}i}$, i.e. $m_{\bar{A}i} \equiv m_{Ai}$. This follows from the CPT-invariance of QED, which ensures that the two-particle scattering amplitude at negative E describes the scattering amplitude of the two antiparticles. The range of the dimensionless radial variable $\tilde{r}$ is $0 < \tilde{r} < \infty$ both for atoms and for antiatoms. In the old variable r, antiatoms have negative distances. This throws new light also on the static limit $m_1/m_2 = 0$. Here one defines $E_e = E - m_2$ as the electron energy. For $E < 0$, one may use $E_e = E + m_2$. From the static Dirac equation in the variable $\tilde{r}$, one obtains a spectrum which is symmetric around $E = 0$ [5], for $V = -\alpha/\tilde{r}$. If one wants to keep this symmetry as a result of CPT also in the case of a finite nuclear charge distribution, one should parameterize $V(\tilde{r})$ rather than $V(r)$.

Of course, the mere CPT-invariance of a spectrum does not guarantee its correctness. Division of the Dirac-Breit equation by E and reformulation in terms of $\tilde{\mathbf{r}}_1 = \mathbf{r}_1/E$, $\tilde{\mathbf{r}}_2 = \mathbf{r}_2/E$, $\tilde{\mathbf{r}} = \mathbf{r}/E$ also produces a CPT-invariant spectrum. But as the interaction in this case does not reproduce the QED Born approximation at all energies, one may hope that an 8-component formalism is again more successful. The corresponding equation has been presented in (10) and the Born approximation has been given in a suitable form in (33), but some details are still missing. However, it is clear that the 8-component formalism will be quadratic in the external potential V_e, but linear in V.

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APPENDIX A: COMBINATIONS OF THE HYPERFINE OPERATOR WITH $\alpha\mathbf{p}$

Writing $\boldsymbol{\sigma}_1\mathbf{p}$ as $\frac{1}{2}(\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma})\mathbf{p}$, one observes from (21) that $\Delta\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^\times$ transform triplets χ_t into the singlet χ_s and vice versa. As a result, the combination required in (5) may be written as:

$$\frac{1}{2}\mathbf{p}\Delta\boldsymbol{\sigma} - iV\mathbf{p}\boldsymbol{\sigma}^\times/E = \left[\frac{1}{2} + (P_t - P_s)V/E\right]\mathbf{p}\Delta\boldsymbol{\sigma}\,. \tag{A1}$$

For total angular momentum f, the triplet states with $l = f$ are excluded from $\mathbf{p}\Delta\boldsymbol{\sigma}$ by parity conservation. Thus one has in the notation of [4]:

$$\tilde{\mathbf{p}}\Delta\boldsymbol{\sigma} = 2i\begin{pmatrix} 0 & 0 & 0 & \partial_- \\ 0 & 0 & 0 & -F/r \\ 0 & 0 & 0 & 0 \\ \partial_+ & F/r & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{p}}\boldsymbol{\sigma}^\times = 2\begin{pmatrix} 0 & 0 & 0 & -\partial_- \\ 0 & 0 & 0 & F/r \\ 0 & 0 & 0 & 0 \\ \partial_+ & F/r & 0 & 0 \end{pmatrix}, \tag{A2}$$

with $\tilde{\mathbf{p}} = r\mathbf{p}/r$, $\partial_\pm = \partial_r \pm 1/r$ and $F = \sqrt{f(f+1)}$.

APPENDIX B: CONNECTION WITH THE FORM DERIVED FROM BREIT OPERATORS

We substitute in (5) $\psi = e^x \psi_B$ and multiply the equation by e^{-x} from the left, where the operator x is of the order of V/E and commutes with γ_5 and β. To order α^4, one may then approximate:

$$e^{-x}\,\mathbf{p}\boldsymbol{\alpha}\,e^{x} \approx (1-x)\,\mathbf{p}\boldsymbol{\alpha}\,(1+x) \approx \mathbf{p}\boldsymbol{\alpha} + [\mathbf{p}\boldsymbol{\alpha}, x]\,, \quad e^{-x}\,K_I/E\,e^{x} \approx K_I/E\,. \tag{B1}$$

Choosing now

$$x = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 V/2E \tag{B2}$$

and extracting a common factor γ_5, one has:

$$e^{-x}\,(\mathbf{p}\boldsymbol{\sigma}_1 - iV\mathbf{p}\boldsymbol{\sigma}^{\times}/E)\,e^{x} = \mathbf{p}\boldsymbol{\sigma}_1 - \frac{1}{2E}(i\boldsymbol{\sigma}^{\times} - \boldsymbol{\sigma}_2)[V,\mathbf{p}]\,. \tag{B3}$$

The second piece is the hyperfine operator derived from Breit operators [4] and is known to reproduce the hyperfine structure of leptonium to order α^4, including positronium [5]. It can be rewritten in compact form:

$$-\frac{1}{2E}(\boldsymbol{\sigma}^{\times} + i\boldsymbol{\sigma}_2)[V,\boldsymbol{\nabla}] = \frac{i}{2E}[\boldsymbol{\sigma}_1\boldsymbol{\nabla}, V]\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\,. \tag{B4}$$

APPENDIX C: SPIN SUMMATION

The propagator of a lepton-antilepton pair will be needed in the perturbative interaction with radiation. Here we merely perform the spin summation for the trivial case of a free pair. We remind the reader that in the 16-component formalism, one defines for particles $i = 1, 2$:

$$\gamma_i^0 = \beta_i\,, \quad \boldsymbol{\gamma}_i = \gamma_i^0\boldsymbol{\alpha}_i = \gamma_i^0\gamma_{5i}\boldsymbol{\sigma}_i\,, \quad \not{p}_i = p_i\gamma_i\,, \tag{C1}$$

which leads to the following form of the spin summation:

$$\sum_{\text{spins}} u_1 u_2 \bar{u}_1 \bar{u}_2 = (\not{p}_1 + m_1)(\not{p}_2 + m_2)\,. \tag{C2}$$

A similar notation may also be used in the 8-component version, but with the understanding $\gamma_1^0 = \gamma_2^0 = \beta\,,\ \gamma_{51} = \gamma_{52} = \gamma_5$. Consequently, $\not{p}_1\not{p}_2 \neq \not{p}_2\not{p}_1$, but:

$$\not{p}_1\beta\not{p}_2 = \not{p}_2\beta\not{p}_1\,, \quad \not{p}_1\not{p}_2\beta = \beta\not{p}_2\not{p}_1\,. \tag{C3}$$

The free leptonium equation (10) and the corresponding equation for w^+ become:

$$(m_2\beta\not{p}_1 - m_1\not{p}_2)u = 0\,, \quad w^+(m_2\beta\not{p}_1 - m_1\not{p}_2) = 0\,. \tag{C4}$$

From (C3) and (C4), one easily verifies the following spin summation:

$$s = \sum_{\text{spins}} u w^{\dagger} = m_2\not{p}_1\beta + m_1\not{p}_2\,. \tag{C5}$$

It is remarkable that this 8×8-matrix is linear in $\not{p}_1$ and $\not{p}_2$, while the 16×16-matrix (C2) also contains $\not{p}_1\not{p}_2$.

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