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A Manifestly Gauge-Invariant Approach to Charged Particles [1]

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Abstract

In this article we provide a manifestly gauge-invariant approach to charged particles. It involves (1) Green functions of gauge-invariant operators and (2) Feynman rules which do not depend on any kind of gauge-fixing condition. First, we provide a thorough analysis of QED. We propose a specific set of gauge-invariant Green functions and describe a manifestly gauge-invariant technique to evaluate them. Furthermore we show how Green's functions and Feynman rules of the manifestly gauge-invariant approach and of the Faddeev-Popov ansatz are related to each other. Finally, we extend this manifestly gauge-invariant approach to the Standard Model of electroweak interactions.

Motivation for such an approach is abundant. A gauge-dependent framework does obstruct not only theoretical insight but also phenomenological analyses of precision experiments at LEP. Unresolved issues include the analysis of finite width effects of unstable particles and of oblique corrections parametrised by, e.g., S, T, and U. Quite another example of considerable complications caused by a gauge-dependent framework is the matching of a full and an effective theory, relevant for effective field theory analyses of the symmetry-breaking sector of the Standard Model.

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1 Introduction

Almost every analysis of gauge theories involves a manifestly gauge-dependent technique based on gauge-dependent Green functions. This gauge-dependence, however, manifests itself only in the off-shell behaviour of these functions. Their pole positions and residues, i.e., particle masses, decay widths, and S-matrix elements, are gauge-independent. Due to this property such functions are interesting and useful objects for physicists.

Naively this appears to be all there is to say. As long as the results are gauge-independent there seems to be no reason at all why the symmetry should be manifest throughout the whole calculation. The great success with which gauge-dependent techniques are employed seems indeed to support this notion.

Real life, however, is a more complicated matter. The literature contains quite a number of examples [1, 2, 3, 4, 5, 6] where the gauge-dependence of the perturbative approach does obstruct not only theoretical insight but also phenomenological analyses of precision experiments at presently accessible energies.

Several attempts [1, 2, 3, 6, 7, 8] have recently been made in order to resolve these issues. The techniques employed differ in the degree with which symmetry properties are manifest. The important point, however, is that they all are still gauge-dependent.

The purpose of this article is to provide a manifestly gauge-invariant approach to analyse gauge theories with charged particles. It does not involve any kind of gauge-dependence. Its two main ingredients are (1) Green functions of gauge-invariant operators and (2) Feynman rules which do not involve any kind of gauge fixing. Some aspects of this approach have already been discussed in the literature [9, 10] where it has been applied to specific problems. Some important issues are, however, still unresolved. They include the particular treatment of charged particles which we will focus on in this work. Before going into detail, we would like to recall what kind of problems are encountered in a gauge-dependent approach and why this is relevant to phenomenological analyses.

Oblique corrections describe the effects of new physics on the vacuum polarisation of the Standard Model gauge-bosons currently measured with precision electroweak experiments [11]. Since these corrections involve off-shell properties of the corresponding Green functions it is by no means clear how to provide a physical, i.e., gauge-independent definition of these quantities within the usual gauge-dependent framework. As a matter of fact, bosonic contributions to the relevant self-energies turn out to be gauge-dependent already at the one-loop level [2]. On the other hand, any quantity which is intended to parametrise new physics' effects of a theory yet unknown should obviously be free of any gauge artefacts.

Problems of this kind arise whenever certain gauge-dependent contributions cannot cancel each other order by order in perturbation theory. Other examples include the analyses of finite-width effects for unstable particles [1] or of a finite gluon mass [3] which involve the re-summation of certain subsets of Feynman diagrams. As a result physical quantities turn out to depend on the gauge-fixing parameter.

One attempt to remove this problem is the so called pinch technique [3, 7]. It sticks to the inherently gauge-dependent framework, yet adds a number of rules about how to modify 1PI functions such that the gauge-parameter dependence of physical quantities disappears.

The background field method [8] is another attempt. Through a particular choice of gauge-fixing condition the background field action turns out to be invariant under gauge transformations of the background fields. This property of the effective action may indeed be quite useful in certain applications. Nevertheless, the background field method deals with gauge-dependent Green functions.

Both approaches fail to resolve the problems described above. This may be illustrated with the attempts to provide a gauge-independent parametrisation of oblique corrections. It was mentioned above that bosonic contributions to the relevant self-energies are gauge-dependent if perturbation theory is applied in the straightforward fashion. Both, the pinch technique and the background field method were used to further investigate this problem.

The attempt based on the pinch technique yields a result for bosonic one-loop corrections which does not depend on the gauge-fixing parameter [2]. The approach based on the background field method, on the other hand, yields corrections which still depend on the gauge-fixing parameter [4]. For the particular choice $\xi = 1$ both results agree.

First of all, this reminds us of the fact that independence of the gauge-fixing parameter is only a necessary condition for gauge-independence, not a sufficient one. Moreover, it demonstrates that this handle does not help to pin down the physical contents of oblique corrections, analysed within a gauge-dependent framework. At present this must still be considered an unresolved issue.

Due to the finite width of the W-boson gauge invariance is also an issue for the analysis of W-pair production at LEP. A detailed account of the problems of various gauge-dependent approaches can be found in the report of the LEP working group [1].

Effective field theories provide another example of unwanted complications caused by a gauge-dependent framework. The low-energy structure of a theory containing light and heavy particle species which are separated by a mass gap can adequately be described by an effective theory which contains only the light particles. The Standard Model with a heavy Higgs boson is an example which has recently received considerable attention [12]. In order to determine the effective Lagrangian, which describes the effective field theory, one requires that corresponding Green functions in both theories have the same low-energy structure. If a gauge-dependent framework is used [5, 6] one again has to make sure that no gauge artefacts enter the low-energy constants of the effective Lagrangian. A detailed discussion of the complications involved with matching gauge-dependent Green functions may be found in Ref. [9].

To avoid these problems one should match only gauge-invariant quantities, such as S-matrix elements [6]. As it turns out, however, matching S-matrix elements is quite cumbersome. Functional techniques [13], which involve Green's functions, are much easier to use. A manifestly gauge-invariant technique combines both advantages: The absence of gauge-artefacts and the simplicity and elegance of the functional approach [9].

There certainly is ample motivation to provide a manifestly gauge-invariant framework to analyse gauge theories. This work focuses on the particulars of charged particles. In addition to formulating this approach we also compare it thoroughly with the usual approach based on the Faddeev-Popov ansatz.

The present article is organised as follows: In the next section we provide a thorough analysis of a manifestly gauge-invariant approach to QED. In particular, we propose a specific set of gauge-invariant Green functions and describe a manifestly gauge-invariant technique to evaluate them. Feynman rules are summarised at the end of the section. Furthermore we show how Green's functions and Feynman rules of this approach and of the Faddeev-Popov ansatz are related to each other. Section 3 contains a detailed discussion of the results. In Section 4 we extend this manifestly gauge-invariant approach to the Standard Model of electroweak interactions. Again we compare the Green functions of this approach with those of the Faddeev-Popov ansatz. Finally, we summarise our results in Section 5.

2 QED

We consider Green's functions of the fermion field ψ and the electromagnetic field strength $F_{\mu\nu}$. If there were no gauge symmetry these Green functions would be generated by the vacuum-to-vacuum transition amplitude

$$e^{iW[\eta,\bar{\eta},k_{\mu\nu}]} = \langle 0_{\rm out}|0_{\rm in}\rangle_{\eta,\bar{\eta},k_{\mu\nu}} = \langle 0|T\left[e^{i\bar{\eta}\psi_{\rm op}+i\bar{\psi}_{\rm op}\eta+ik_{\mu\nu}F^{\mu\nu}_{\rm op}}\right]|0\rangle\ . \qquad (1)$$

However, this amplitude is associated with the Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{QED} + \bar{\eta}\psi + \bar{\psi}\eta + k_{\mu\nu}F^{\mu\nu} && (2)\\ \mathcal{L}_{QED} &= \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} && (3) \end{aligned}$$

whose source-independent piece $\mathcal{L}_{QED}$ is invariant under gauge transformations of the form

$$\begin{aligned} \psi &\to e^{-i\omega}\psi\ , && (4)\\ A_\mu &\to A_\mu + \partial_\mu\omega\ . && (5) \end{aligned}$$

The covariant derivative is given by

$$D_\mu = \partial_\mu + iA_\mu\ . \qquad (6)$$

The external source η, on the other hand, is a fixed function of space-time. Hence, Green's functions involving the fermion fields ψ and $\bar{\psi}$ do not reflect the symmetry properties of the Lagrangian $\mathcal{L}_{QED}$. What is worse, the generating functional given in Eq. (1) is not even well-defined. The general problem associated with gauge symmetries is readily worked out in the path-integral representation of the generating functional, which is of the form

$$e^{iW[\eta,\bar{\eta},k_{\mu\nu}]} = \int d\mu[\psi,\bar{\psi},A_\mu]e^{i\int {\rm d}^{\rm d}x\, \mathcal{L}_{QED}+\bar{\eta}\psi+\bar{\psi}\eta+k_{\mu\nu}F^{\mu\nu}}\ . \qquad (7)$$

At leading order in the loop expansion the generating functional is given by the classical action

$$W[\eta,\bar{\eta},k_{\mu\nu}] = \int \mathrm{d}^{\mathrm{d}}x\,\mathcal{L}_{QED}(\psi^{\mathrm{cl}},A_\mu^{\mathrm{cl}}) + \bar{\eta}\psi^{\mathrm{cl}} + \bar{\psi}^{\mathrm{cl}}\eta + k^{\mu\nu}F_{\mu\nu}^{\mathrm{cl}}\ . \tag{8}$$

The classical fields A_μ^{cl} and ψ^{cl} are determined by the condition that the phase of the integrand in Eq. (7) be stationary, i.e., that

$$\delta\int \mathrm{d}^{\mathrm{d}}x\,\mathcal{L}_{QED} + \bar{\eta}\psi + \bar{\psi}\eta + k_{\mu\nu}F^{\mu\nu} = 0\ . \tag{9}$$

Since only the source terms involving the external field η violate the gauge symmetry, variations of the fields corresponding to infinitesimal gauge transformations yield the condition

$$\bar{\eta}\psi^{\mathrm{cl}} - \bar{\psi}^{\mathrm{cl}}\eta = 0\ . \tag{10}$$

It is a concise expression for the problem of the amplitude in Eq. (1). On the one hand, gauge invariance wants the photon to couple to conserved currents only. This requires condition (10) to be satisfied. The external source η, on the other hand, does not want to do us this favour. The solutions of the equations of motion do not satisfy this condition. It is, in fact, obvious that the electromagnetic current cannot be conserved in the presence of sources which create or destroy charged particles. In other words, the generating functional $W[\eta,\bar{\eta},k_{\mu\nu}]$ is not properly defined.

The usual remedy is to abandon gauge invariance. A supplementary condition of the form

$$f(A_\mu) = 0 \tag{11}$$

is introduced, which freezes the gauge degree of freedom. Green's functions are then defined through the Faddeev-Popov ansatz [14] for the generating functional, i.e., through

$$e^{iW_f[\eta,\bar{\eta},k_{\mu\nu}]} \doteq \int d\mu[\psi,\bar{\psi},A_\mu]\,\det(M_f)\,\delta\left(f(A_\mu)\right)\, e^{i\int \mathrm{d}^{\mathrm{d}}x\,\mathcal{L}_{QED}+\bar{\eta}\psi+\bar{\psi}\eta+k_{\mu\nu}F^{\mu\nu}}\ . \tag{12}$$

The matrix M_f is the variation of the functional f with respect to a gauge transformation. This ansatz defines gauge-dependent Green functions.

There is, however, another solution to the problem. Note that the source term involving the external field $k_{\mu\nu}$ does not affect condition (10). This follows from the gauge invariance of the field strength. Hence one may also define Green's functions through the vacuum-to-vacuum transition amplitude

$$e^{i\bar{W}[N,\bar{N},k_{\mu\nu}]} = \langle 0_{\rm out}|0_{\rm in}\rangle_{N,\bar{N},k_{\mu\nu}} = \langle 0|T\left[e^{i\bar{N}\psi_{\rm op}+i\bar{\psi}_{\rm op}N+ik_{\mu\nu}F^{\mu\nu}_{\rm op}}\right]|0\rangle \tag{13}$$

which involves a gauge-variant external source N that transforms according to

$$N \to e^{-i\omega}N \tag{14}$$

under gauge transformations. Sources of this form do not couple to the gauge degree of freedom and, thus, define gauge-invariant Green functions. Problem (10) does not appear. We will see below how this comes about.

From a physical point of view the second approach makes even more sense. After all, gauge invariance means that the phase of the fermion field is completely arbitrary at any space-time point – and physically unobservable. Hence, any artificial dependence on the gauge degree of freedom should be expected to do more harm than good.

The path-integral representation of the amplitude (13) is of the form

$$e^{i\bar{W}[N,\bar{N},k_{\mu\nu}]} = \int d\mu[\psi,\bar{\psi},A_\mu]e^{i\int \mathrm{d^d}x\,\mathcal{L}_{QED}+\bar{N}\psi+\bar{\psi}N+k_{\mu\nu}F^{\mu\nu}} \ . \tag{15}$$

Note that no gauge-fixing condition is introduced, in contrast to the ansatz in Eq. (12). In the next subsection we will evaluate this functional for a specific external source N.

Beforehand, we should add some remarks about gauge fixing. The vacuum-to-vacuum transition amplitude (1) and the corresponding path-integral representation (7) are not well-defined because the external source η couples to the gauge degree of freedom. In order to define Green's functions properly in the presence of such sources one has to fix the gauge and work with the Faddeev-Popov ansatz (12).

The situation is quite different if only gauge-invariant sources are considered. In this case one may also fix the gauge in order to evaluate the path integral. However, one need not do so. Both ways yield the same Green functions. For example, the generating functional $\bar{W}[0,0,k_{\mu\nu}]$ defined in Eq. (15) is exactly the same as the generating functional $W_f[0,0,k_{\mu\nu}]$ defined in Eq. (12). The difference is only seen at intermediate steps of a calculation. If the gauge is fixed, contributions from certain diagrams depend on the gauge-fixing parameter ξ. In contrast to the Faddeev-Popov ansatz with gauge-dependent sources, however, this dependence drops out order by order in perturbation theory if all contributions to a given Green function are combined – even off-shell.

Thus, if one is interested in a specific subset of contributions only, like, for instance, oblique corrections, it is better to evaluate the generating functional without gauge fixing. Otherwise one will still encounter the complications one actually wants to avoid, even if gauge-invariant Green functions are considered. This is also the case for the analysis of finite width effects of unstable particles.

Note that this remarks do not only apply to the specific case we are considering here, but are valid in any gauge theory, including the Standard Model of electroweak interactions and QCD.

2.1 QED the manifestly gauge-invariant way

It is not difficult to see that sources with transformation properties as specified in Eq. (14) do exist. The most straightforward construction may be of the form

$$N(x) = \eta(x) e^{-i\int_{-\infty}^{x} \mathrm{d}x'_\mu A^\mu(x')} \; , \tag{16}$$

which can already be found in Ref. [15]. As long as one is only interested in the existence of such sources and some of their formal properties, this choice will surely suffice. However, the corresponding Green functions have unwanted properties. In particular, they depend on the specific paths used to evaluate the integrals in Eq. (16) for each external leg. Furthermore, this source depends on the dynamical component of the gauge field.

Another source term with the same transformation properties is of the form

$$N(x) = \eta(x) e^{-i\int \mathrm{d}^d y G_0(x-y)\partial_\mu A^\mu(y)} \; , \tag{17}$$

where the propagator G_0 is defined as

$$\Box G_0(z) = \delta^{(d)}(z) \; . \tag{18}$$

It does not suffer from the path-dependence the other source has. And it does not involve any dynamical degree of freedom. This is exactly the source whose properties we will investigate here. Thus, from now on, the external field N is of the particular form given in Eq. (17).

We define Green's functions involving the fermion field through derivatives of the generating functional (15) with respect to the external field η. They differ by a phase from corresponding derivatives with respect to the external source N. To make this definition more explicit we redefine the generating functional as

$$e^{iW[\eta,\bar{\eta},k_{\mu\nu}]} \doteq \int d\mu[\psi,\bar{\psi},A_\mu] e^{i\int \mathrm{d}^{\mathrm{d}}x\,\mathcal{L}_{QED}+\bar{N}\psi+\bar{\psi}N+k_{\mu\nu}F^{\mu\nu}} \tag{19}$$

for the source N given in Eq. (17).

It is not quite straightforward to write down Feynman rules to evaluate the functional given in Eq. (19). First of all, we do not want to introduce any gauge-fixing condition in the path integral for reasons described above. Hence, the usual definition of the gauge-field propagator is of no avail. Furthermore, the source N in Eq. (17) involves an exponential factor which, one would think, should produce vertices that emit an arbitrary number of photons. This presumption, however, will turn out to be wrong.

Thus, in order to provide a sound understanding of our manifestly gauge-invariant approach to Green's functions, we will go step by step through the loop expansion. Feynman rules, which we think will turn out to be much simpler than anticipated, are given at the end of this section.

Trees

At leading order in the loop expansion the generating functional (19) is given by the classical action

$$W[\eta,\bar{\eta},k_{\mu\nu}] = \int \mathrm{d}^{\mathrm{d}}x\,\mathcal{L}_{QED}(A_\mu^{\mathrm{cl}},\psi^{\mathrm{cl}}) + \bar{N}\psi^{\mathrm{cl}} + \bar{\psi}^{\mathrm{cl}}N + k^{\mu\nu}F_{\mu\nu}^{\mathrm{cl}} \tag{20}$$

where the fields A_μ^{cl} and ψ^{cl} satisfy the equations of motion

$$\begin{aligned} (i\not{D}-m)\,\psi^{\mathrm{cl}} &= -N\ , & (21)\\ \bar{\psi}^{\mathrm{cl}}\left(i\overleftarrow{\not{D}}-m\right) &= -\bar{N}\ , & (22)\end{aligned}$$

and

$$\partial^\mu F_{\mu\nu}^{\mathrm{cl}} = 2e^2\partial^\mu k_{\mu\nu} + e^2\bar{\psi}^{\mathrm{cl}}\gamma_\nu\psi^{\mathrm{cl}} + ie^2\partial_\nu\frac{1}{\Box}\left(\bar{N}\psi^{\mathrm{cl}} - \bar{\psi}^{\mathrm{cl}}N\right)\ , \tag{23}$$

with

$$\overleftarrow{D}_\mu = -\overleftarrow{\partial}_\mu + iA^{\rm cl}_\mu \ . \tag{24}$$

The third term on the right hand side of Eq. (23) is due to the variation of the phase in Eq. (17). Its effect is readily worked out. Eqs. (21) and (22) imply the identity

$$\partial_\mu \left(\bar\psi^{\rm cl}\gamma^\mu\psi^{\rm cl}\right) = i\left(\bar\psi^{\rm cl}N - \bar N\psi^{\rm cl}\right) \ . \tag{25}$$

It shows that the electromagnetic current is not conserved in the presence of an external source which emits or absorbs charged particles (cf. problem (10)). Gauge invariance, on the other hand, requires that the photon field couples to conserved currents only. The third term on the right hand side of Eq. (23) ensures that this is indeed the case. Using Eq. (25) one obtains

$$\partial^\mu F^{\rm cl}_{\mu\nu} = 2e^2\partial^\mu k_{\mu\nu} + e^2 {\rm PT}_{\nu\rho}\bar\psi^{\rm cl}\gamma^\rho\psi^{\rm cl} \ , \tag{26}$$

where

$${\rm PT}_{\mu\nu} = g_{\mu\nu} - {\rm PL}_{\mu\nu} \quad \text{and} \quad {\rm PL}_{\mu\nu} = \partial_\mu \frac{1}{\Box}\partial_\nu \tag{27}$$

project onto the transversal and longitudinal degrees of freedom respectively. If the external source η is switched off the electromagnetic current is conserved and the transversal projector in Eq. (26) reduces to the identity.

Gauge invariance implies that the equations of motion have a whole class of solutions. Every two representatives are related to each other by a gauge transformation. In order to solve these equations we write the fermion field as (cf. Eq. (17))

$$\psi^{\rm cl}(x) = \chi(x)e^{-i\int {\rm d}^{\rm d}y G_0(x-y)\partial^\mu A^{\rm cl}_\mu(y)} \ . \tag{28}$$

As a result, the phase drops out of Eq. (21), which takes the form

$$\left(i\not{D}^T - m\right)\chi = -\eta \ . \tag{29}$$

It involves only the transversal component $A^T_\mu \doteq {\rm PT}_{\mu\nu}A^\nu$ of the gauge field through the derivative

$$D^T_\mu \doteq \partial_\mu + iA^T_\mu \ . \tag{30}$$

The longitudinal component $A^L_\mu \doteq \mathrm{PL}_{\mu\nu} A^\nu$ does not enter Eq. (26) either. Hence, the solutions of the equations of motion are of the form

$$
\begin{aligned}
A^{L,\mathrm{cl}}_\mu &= \partial_\mu \omega \ , && (31)\\
\psi^{\mathrm{cl}} &= e^{-i\omega}\chi \ , && (32)\\
A^{T,\mathrm{cl}}_\mu(x) &= ie^2 \int \mathrm{d}^{\mathrm{d}}y \Delta_{\mu\nu}(x-y)\left(2\partial_\rho k^{\rho\nu}(y) + (\bar{\chi}\gamma^\nu\chi)(y)\right) \ , && (33)\\
\chi(x) &= i\int \mathrm{d}^{\mathrm{d}}y S(x-y)\left(-\eta(y) + \left(\not{A}^{T,\mathrm{cl}}\chi\right)(y)\right) \ , && (34)
\end{aligned}
$$

with an undetermined function ω describing the gauge degree of freedom. It is important to note that this function does not enter the generating functional in Eq. (19). Gauge invariance ensures that it drops out. Hence, Green's functions do not depend on this function either. This justifies the redefinition of the generating functional in Eq. (19). The free propagators are defined as

$$
\begin{aligned}
\Delta_{\mu\nu}(x-y) &= i\langle x|\frac{\mathrm{PT}_{\mu\nu}}{-\Box}|y\rangle \ , && (35)\\
S(x-y) &= i\langle x|\frac{-1}{i\not{\partial}-m}|y\rangle \ . && (36)
\end{aligned}
$$

Gauge invariance is indeed manifest.

One Loop

A convenient way to evaluate the one-loop contribution to the generating functional (19) is the method of steepest descent. Using the parametrisation

$$
\begin{aligned}
A_\mu &= A^{\mathrm{cl}}_\mu + \sqrt{2}eq_\mu && (37)\\
\psi &= \psi^{\mathrm{cl}} + \kappa && (38)
\end{aligned}
$$

for the quantum fluctuations one obtains for the one-loop approximation

$$
\begin{aligned}
& e^{iW[\eta,\bar{\eta},k_{\mu\nu}]} && (39)\\
&= e^{i\int \mathrm{d}^{\mathrm{d}}x\, \mathcal{L}(A^{\mathrm{cl}}_\mu,\psi^{\mathrm{cl}})} \int d\mu[\kappa,\bar{\kappa},q_\mu] e^{i\int \mathrm{d}^{\mathrm{d}}x\, \bar{\kappa}D_{\bar{\kappa}\kappa}\kappa + \bar{\kappa}D^\nu_{\bar{\kappa}q}q_\nu + q_\mu D^\mu_{q\kappa}\kappa + q_\mu D^{\mu\nu}_{qq}q_\nu} \ ,
\end{aligned}
$$

where

$$
\mathcal{L} = \mathcal{L}_{QED} + \bar{N}\psi + \bar{\psi}N + k_{\mu\nu}F^{\mu\nu} \ , \qquad (40)
$$

$$
\begin{aligned}
D_{\bar{\kappa}\kappa} &= i\not{D} - m \;, && (41)\\
D^{\nu}_{\bar{\kappa}q} &= -\sqrt{2}e\left(\gamma^{\nu}\psi^{\rm cl} + iN\frac{1}{\Box}\partial^{\nu}\right) \;, && (42)\\
D^{\mu}_{q\kappa} &= -\sqrt{2}e\left(\bar{\psi}^{\rm cl}\gamma^{\mu} + i\partial^{\mu}\frac{1}{\Box}\bar{N}\right) \;, && (43)\\
D^{\mu\nu}_{qq} &= \Box g^{\mu\nu} - \partial^{\mu}\partial^{\nu} + e^2\partial^{\mu}\frac{1}{\Box}\left(\bar{N}\psi^{\rm cl} + \bar{\psi}^{\rm cl}N\right)\frac{1}{\Box}\partial^{\nu} \;. && (44)
\end{aligned}
$$

All terms involving the free massless propagator $\Box^{-1}$ again arise from the variation of the phase in Eq. (17). We proceed by diagonalizing the quadratic form in the exponent of Eq. (39) with the shift

$$
\kappa \to \kappa + \sqrt{2}e(i\not{D} - m)^{-1}\left(\not{q}\psi^{\rm cl} + iN\frac{1}{\Box}\partial_{\mu}q^{\mu}\right) \tag{45}
$$

and obtain the diagonal matrix

$$
\begin{aligned}
D_{\bar{\kappa}\kappa} &= i\not{D} - m \;, && (46)\\
D^{\mu\nu}_{qq} &= \Box g^{\mu\nu} - \partial^{\mu}\partial^{\nu} + e^2\partial^{\mu}\frac{1}{\Box}\left(\bar{N}\psi^{\rm cl} + \bar{\psi}^{\rm cl}N\right)\frac{1}{\Box}\partial^{\nu} && (47)\\
&\quad - 2e^2\left(\bar{\psi}^{\rm cl}\gamma^{\mu} + i\partial^{\mu}\frac{1}{\Box}\bar{N}\right)(i\not{D} - m)^{-1}\left(\gamma^{\nu}\psi^{\rm cl} + iN\partial^{\nu}\frac{1}{\Box}\right) \;.
\end{aligned}
$$

Next we employ the equations of motion (21) and (22) to remove all occurrences of the external fields N and $\bar{N}$. One obtains, for example,

$$
\gamma^{\nu}\psi^{\rm cl} + iN\partial^{\nu}\frac{1}{\Box} = \gamma_{\mu}\psi^{\rm cl}{\rm PT}^{\mu\nu} - i(i\not{D} - m)\psi^{\rm cl}\partial^{\nu}\frac{1}{\Box} \;. \tag{48}
$$

The final result turns out to be

$$
D^{\mu\nu}_{qq} = \Box g^{\mu\nu} - \partial^{\mu}\partial^{\nu} - {\rm PT}^{\mu\alpha}\sigma_{\alpha\beta}{\rm PT}^{\beta\nu} \tag{49}
$$

with

$$
\sigma_{\mu\nu} = 2e^2\bar{\psi}^{\rm cl}\gamma_{\mu}(i\not{D} - m)^{-1}\gamma_{\nu}\psi^{\rm cl} \;. \tag{50}
$$

It is quite remarkable. All contributions from the variation of the source term $\bar{N}\psi + \bar{\psi}N$, which seemed prone to create additional infrared singularities, transformed into the apparent transversal structure of the full propagator (49). In fact, only the usual vertex from the interaction $\bar{\psi}\not{A}\psi$ remains.

Before we introduce a shortcut to obtain this result directly, we have to explain how to evaluate the path integral (39).

Gauge invariance implies that the operator $D_{qq}^{\mu\nu}$ given in Eq. (49) has zero eigenvectors of the form

$$q^L_{\mu,n} = \partial_\mu \omega_n \ . \tag{51}$$

We will assume the scalar functions ω_n to be eigenvectors of the d'Alembert operator, i.e.,

$$-\Box \omega_n = l_n \omega_n \ . \tag{52}$$

The procedure to evaluate path integrals with zero-modes is well known [9, 10]. The expansion of the fluctuation q_μ in terms of eigenvectors of the operator $D_{qq}^{\mu\nu}$ is given by

$$q^\mu = \sum_n a_n \xi_n^\mu + \sum_m b_m \zeta_m^\mu \ , \tag{53}$$

where $\zeta_m^\mu = \partial^\mu \omega_m$ and ξ_n^μ have zero and non-zero eigenvalues, respectively.

In order to evaluate the path integral (39), we use Polyakov's method [16] and equip the space of fields with a metric

$$||q||^2 = \int \mathrm{d}^\mathrm{d} x q_\mu q^\mu \tag{54}$$

$$= \sum_n a_n^2 + \sum_m b_m^2 l_m \ . \tag{55}$$

The volume element associated with this metric is then given by

$$\mathrm{d}\mu[\kappa, \bar{\kappa}, q] = \mathcal{N} \prod_n \mathrm{d}a_n \prod_m \mathrm{d}b_m \sqrt{\det(-\Box)} \ . \tag{56}$$

The integration over the zero-modes yields the volume factor of the gauge group, which can be absorbed by the normalisation of the integral. The remaining integral over the non-zero modes is damped by the usual Gaussian factor. Up to an irrelevant infinite constant one obtains the following result for the one-loop generating functional

$$W[\eta, \bar{\eta}, k_{\mu\nu}] = \int \mathrm{d}^\mathrm{d} x\, \mathcal{L}(A_\mu^{\mathrm{cl}}, \psi^{\mathrm{cl}}) - i \ln \det (i\not{D} - m) + \frac{i}{2} \ln \det{}' (-D_{qq}) \tag{57}$$

where $\det'(-D_{qq})$ is defined as the product of all non-zero eigenvalues of the operator $-D_{qq}$. In the Abelian case the contribution from the measure, given by the determinant in Eq. (56), is trivial. Since zero and non-zero eigenvectors are orthogonal to each other, implying $\partial_\mu \xi_n^\mu = 0$, one furthermore verifies the identity

$$\det{}'(-D_{qq}^{\mu\nu}) = \frac{\det\left(-D_{qq}^{\mu\nu} - \partial^\mu\partial^\nu\right)}{\det(-\Box)} \ . \tag{58}$$

Hence, up to another irrelevant infinite constant one-loop contributions containing gauge-boson propagators are of the form

$$\frac{i}{2}\ln\det{}'\left(-D_{qq}\right) = -\frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\left((\Delta i\sigma)^n\right) \ , \tag{59}$$

with the gauge-boson propagator $\Delta_{\mu\nu}$ given in Eq. (35).

There is only one problem left. Both the fermionic determinant in Eq. (57) as well as the full fermion propagator in Eq. (50) depend on the longitudinal component of the gauge field. This can be traced back to Eq. (38). The quantity κ describes the fluctuation of the gauge-independent part of the fermion field ψ as well as of its phase. To remove this dependence, one may set

$$\kappa(x) \to \kappa(x) e^{-i\int \mathrm{d^d}y G_0(x-y)\partial^\mu A_\mu^{\mathrm{cl}}(y)} \tag{60}$$

in Eq. (39). As a result Eqs. (41–44) involve only the fields χ, η and the derivative D_μ^T. Now the one-loop functional (57) is of the form

$$W[\eta,\bar\eta,k_{\mu\nu}] = \int \mathrm{d^d}x\, \mathcal{L}(A_\mu^{\mathrm{cl}},\psi^{\mathrm{cl}}) - i\ln\det\left(i\not{D}^T - m\right) - \frac{i}{2}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\left((\Delta i\bar\sigma)^n\right) \ , \tag{61}$$

with

$$\bar\sigma_{\mu\nu} = 2e^2\bar\chi\gamma_\mu(i\not{D}^T - m)^{-1}\gamma_\nu\chi \ . \tag{62}$$

More loops

The effect of the phase in Eq. (17) is understood. It ensures that the external source η does not couple to the physically unobservable phase of the fermion field ψ. Its presence, however, seems to manifest itself rather painfully during the evaluation of the generating functional, cf. Eqs.(23, 42-44) and (47). One would expect the problem to get even worse beyond the one-loop level.

However, this is not true. Our way to arrive at the results given in Eqs. (31-34) and (61) was so involved because we did not choose a convenient parametrisation for the quantum fluctuations in Eq. (38). A better choice is already indicated in Eq. (60) where we had to rescale the field κ in order to get rid of the gauge degree of freedom in Eqs. (57) and (50). Note that we also removed the phase of the fermion field $\psi^{\rm cl}$ in Eq. (28). Thus, one would better use the parametrisation

$$\begin{aligned} A_\mu &= A_\mu^{\rm cl} + \sqrt{2} e q_\mu \ , & (63) \\ \psi &= e^{-i(\phi+E)}\left(\chi + \kappa\right) \ , & (64) \end{aligned}$$

with

$$\begin{aligned} \phi(x) &= \int {\rm d}y G_0(x-y)\partial^\mu A_\mu^{\rm cl}(y) \ , & (65) \\ E(x) &= \sqrt{2}e\int {\rm d}y G_0(x-y)\partial^\mu q_\mu(y) \ . & (66) \end{aligned}$$

The transition from Eq. (38) to Eq. (64) can in fact be obtained by a transformation of the variable κ with the determinant of the corresponding Jacobian equal to one. One readily verifies, that this parametrisation directly yields Eqs. (31-34) and (61). The phase ϕ and its variation E drop out of the source term $\bar{N}\psi + \bar{\psi}N$ and of the Lagrangian $\mathcal{L}_{QED}$. The same is true for the longitudinal components of the gauge field $A_\mu^{\rm cl}$ and its fluctuation q_μ. It is now straightforward to derive the following result for the full generating functional (39), including all n-loop contributions:

$$\begin{aligned} & e^{iW[\eta,\bar\eta,k_{\mu\nu}]} & (67) \\ &= e^{i\int {\rm d}^{\rm d}x\,\mathcal{L}(A_\mu^{\rm cl},\psi^{\rm cl})}\int d\mu[\kappa,\bar\kappa,q_\mu]e^{i\int {\rm d}^{\rm d}x\,\bar\kappa D_{\bar\kappa\kappa}\kappa + q_\mu D_{qq}^{\mu\nu}q_\nu} \\ &\qquad \times\left\{1-\frac{1}{2}\left(\int {\rm d}^{\rm d}x\,\mathcal{L}^{[3]}\right)^2+\frac{1}{24}\left(\int {\rm d}^{\rm d}x\,\mathcal{L}^{[3]}\right)^4-\dots\right\} \ , \end{aligned}$$

where

$$\begin{aligned} D_{\bar\kappa\kappa} &= i\not{D}^T - m \ , & (68) \\ D_{qq}^{\mu\nu} &= \Box g^{\mu\nu} - \partial^\mu\partial^\nu - {\rm PT}^{\mu\alpha}\bar\sigma_{\alpha\beta}{\rm PT}^{\beta\nu} \ , & (69) \\ \mathcal{L}^{[3]} &= \sqrt{2}e\Big(\bar\kappa + \sqrt{2}e\bar\chi\not{q}^T(i\not{D}^T-m)^{-1}\Big)\not{q}^T\Big(\kappa+\sqrt{2}e(i\not{D}^T-m)^{-1}\not{q}^T\chi\Big) \ , & (70) \end{aligned}$$

and

$$q^T_\mu = \mathrm{PT}_{\mu\nu} q^\nu \ . \tag{71}$$

Any vertex emits non-zero modes only. The full gauge field propagator entering loop graphs is given by the inverse of the operator $D^{\mu\nu}_{qq}$, restricted to the subspace of non-zero modes. Thus, the evaluation of loop contributions of arbitrary order boils down to the evaluation of Gaussian integrals, for example, of the form

$$2 \int \mathrm{d}\mu[q] q^T_\mu(x) q^T_\nu(y) e^{i \int \mathrm{d}^d x q_\alpha D^{\alpha\beta}_{qq} q_\beta} = i \langle x | - D^{-1\,\prime}_{qq\ \mu\nu} | y \rangle / \sqrt{\det{}' \left(-D_{qq} \right)} \ , \tag{72}$$

where

$$D^{-1\,\prime}_{qq\ \mu\rho} D^{\rho\nu}_{qq} = D^{\nu\rho}_{qq} D^{-1\,\prime}_{qq\ \rho\mu} = \mathrm{PT}^\nu_\mu \ . \tag{73}$$

One readily verifies that

$$\begin{aligned} -iD^{-1\,\prime}_{qq} &= -i\mathrm{PT} \left(\Box - \mathrm{PT} \bar{\sigma} \mathrm{PT} \right)^{-1} \mathrm{PT} \\ &= \sum_{n=0}^{\infty} \left(\Delta i \bar{\sigma} \right)^n \Delta \ , \end{aligned} \tag{74}$$

with the free gauge field propagator $\Delta_{\mu\nu}$ given in Eq. (35). The corresponding full fermion propagator is

$$-i \left(i \not{D}^T - m \right)^{-1} = \sum_{n=0}^{\infty} \left(S i \not{A}^{T,\mathrm{cl}} \right)^n S, \tag{75}$$

with the free fermion propagator S given in Eq. (36).

2.2 QED the old-fashioned way

The following discussion of the Faddeev-Popov ansatz (12) for the generating functional $W_f[\eta, \bar{\eta}, k_{\mu\nu}]$ is based on the gauge condition

$$f(A_\mu) = \partial_\mu A^\mu \ . \tag{76}$$

In this case the determinant of the matrix M_f yields an infinite constant which can be absorbed by the normalisation of the path integral. The delta function can be converted into an exponential factor. One obtains the well known result

$$e^{iW^\xi_f[\eta,\bar{\eta},k_{\mu\nu}]} = \int d\mu[\psi, \bar{\psi}, A_\mu] e^{i \int \mathrm{d}^d x \, \mathcal{L}_{QED} + \mathcal{L}_{GF} + \bar{\eta}\psi + \bar{\psi}\eta + k^{\mu\nu} F_{\mu\nu}} \ , \tag{77}$$

with

$$\mathcal{L}_{GF} = \frac{-1}{2\xi e^2} \left(\partial_\mu A^\mu\right)^2 \ . \tag{78}$$

Since it is straightforward to repeat the whole analysis of the last subsection for the case at hand we merely state the results and compare. The classical fields satisfy the equations of motion

$$\begin{aligned} (i\not{D} - m)\, \psi^{\mathrm{cl}} &= -\eta & (79)\\ \partial^\mu F^{\mathrm{cl}}_{\mu\nu} + \frac{1}{\xi}\partial^\mu \partial_\nu A^{\mathrm{cl}}_\mu &= 2e^2 \partial^\mu k_{\mu\nu} + e^2 \bar{\psi}^{\mathrm{cl}} \gamma_\nu \psi^{\mathrm{cl}} & (80)\end{aligned}$$

to be compared with Eqs. (21) and (26). In this case the electromagnetic current is not conserved either. Here however its divergence does enter the equations of motion. It determines the gauge degree of freedom. Due to the presence of the gauge-dependent piece on the left hand side of Eq. (80) problem (10) does not occur here. The solutions are

$$\begin{aligned} A^{\mathrm{cl},\xi}_\mu(x) &= ie^2 \int d^d y \Delta^\xi_{\mu\nu}(x-y) \left(2\partial_\rho k^{\rho\nu}(y) + \left(\bar{\psi}^{\mathrm{cl},\xi} \gamma^\nu \psi^{\mathrm{cl},\xi}\right)(y)\right) \ , & (81)\\ \psi^{\mathrm{cl},\xi}(x) &= i \int \mathrm{d}^{\mathrm{d}} y S(x-y) \left(-\eta(y) + \left(\not{A}^{\mathrm{cl},\xi} \psi^{\mathrm{cl},\xi}\right)(y)\right) \ , & (82)\end{aligned}$$

corresponding to Eqs. (31-33) and (34). They determine the gauge degree of freedom as well. The gauge-field propagator $\Delta^\xi_{\mu\nu}$ is of the form

$$\Delta^\xi_{\mu\nu}(x-y) = i \langle x | \frac{\mathrm{PT}_{\mu\nu}}{-\Box} + \xi \frac{\mathrm{PL}_{\mu\nu}}{-\Box} | y \rangle \ , \tag{83}$$

corresponding to Eq. (35). The fermion propagator is given in Eq. (36). In the limit $\xi \to 0$ (Landau gauge) one obviously recovers the results given in Eqs. (31-33) and (34) for the special case $\omega = 0$. In this case, the longitudinal component of the gauge field vanishes.

Contributions from loops work out in the same way. Since the gauge is fixed, zero-modes do not occur and the evaluation of the path integral corresponding to Eq. (67) is straightforward. Instead of the quantities in Eqs. (68, 69) and (70) it involves

$$\begin{aligned} D^\xi_{\bar{\kappa}\kappa} &= i\not{D} - m \ , & (84)\\ D^{\xi,\mu\nu}_{qq} &= \Box g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu - \sigma_{\alpha\beta} \ , & (85)\\ \mathcal{L}^{\xi[3]} &= \sqrt{2} e \Big(\bar{\kappa} + \sqrt{2} e \bar{\psi}^{\mathrm{cl},\xi} \not{q} (i\not{D} - m)^{-1}\Big) \not{q} \Big(\kappa + \sqrt{2} e (i\not{D} - m)^{-1} \not{q} \psi^{\mathrm{cl},\xi}\Big) & (86)\end{aligned}$$

The one-loop contribution corresponding to the result in Eq. (61) is of the form

$$-i \ln \det \left(i \not{D} - m \right) - \frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr} \left((\Delta^{\xi} i\sigma)^{n} \right) \ . \tag{87}$$

In Landau gauge both are the same.

Contributions beyond one-loop order involve the full gauge field propagator $-iD^{\xi}_{qq}{}^{-1}$. In the limit $\xi \to 0$ it reduces to Eq. (74). Hence, in Landau gauge longitudinal fluctuations of the gauge field do not contribute either.

2.3 Feynman rules

The Feynman rules to evaluate the gauge-invariant generating functional $W[\eta, \bar{\eta}, k_{\mu\nu}]$ given in Eq. (19) are exactly the same as those to evaluate the gauge-dependent generating functional $W_f^{\xi=0}[\eta, \bar{\eta}, k_{\mu\nu}]$ defined in Eq. (77) in Landau gauge. Corresponding Green functions are identical.

This includes a photon mass which might be introduced as an infrared regulator before some of the external momenta are set on the mass-shell. The mass term

$$\frac{1}{2} \mu^2 A_\mu A^\mu \tag{88}$$

which is usually introduced in Eq. (77) is not gauge invariant. Hence, in order to preserve gauge symmetry, one has to use the non-local mass term

$$\frac{1}{2} \mu^2 A^T_\mu A^{T\mu} \tag{89}$$

in Eq. (19) instead, which involves only the transverse components of the gauge field. Again one obtains the same Feynman rules.

Note that off-shell momenta already provide an infrared regulator. Thus, our analysis presented above was not obstructed by infrared problems, since it was concerned with Green's functions only.

3 Discussion

In the following discussion of QED we will adopt a language that equally applies to the non-Abelian case. Thus, we will talk of zero- and nonzero-modes, instead of longitudinal and transversal components.

There are two approaches to define Green's functions for charged particles. One involves a non-local source of the form given in Eq. (17) which defines gauge-invariant Green functions. The other involves gauge-dependent sources and is based on the Faddeev-Popov ansatz in Eq. (12); in this case Green's functions depend on both, the gauge condition and the gauge-fixing parameter ξ. Surprisingly, both approaches yield exactly the same Green functions in Landau gauge, i.e., for the gauge condition (76) and in the limit $\xi \to 0$.

The quadratic form in the exponent of Eq. (67) has zero-modes which is a consequence of gauge invariance. These zero-modes correspond to fluctuations of the fields that are gauge transformations. This divides the space of all fluctuations into two parts. The subspace of all nonzero-modes is the physical one. The essential feature of the manifestly gauge-invariant approach is that it is restricted to this subspace only. In the equations of motion (21) and (26) the gauge degree of freedom drops out. One-loop corrections involve the determinant of a differential operator defined on the subspace of nonzero-modes. The inverse of the same operator, i.e., the full propagator, is also defined on this subspace. In Eq. (67) the integration over the zero-modes yields the volume factor of the gauge group and is absorbed by the normalisation of the integral. This explains why the exponential factor of the source in Eq. (17) does not produce any vertex. It involves only the longitudinal component of the gauge field which does not affect the dynamics of the theory. The other source in Eq. (16) is a different matter. It also involves the physical components of the gauge field.

If a gauge-fixing condition is introduced, like in Eq. (12), all quantities are extended to the full space. All modes propagate (see the propagator in Eq. (83)). In the case of gauge-invariant sources, the unphysical contributions from the subspace of zero-modes are exactly cancelled order by order by the corresponding contributions from ghosts, which are trivial in the Abelian case. This cancellation does not happen if the sources couple to the gauge degree of freedom; the resulting Green functions are gauge-dependent.

In the limit $\xi \to 0$ the exponential factor in Eq. (77) involving the gauge-fixing part of the Lagrangian becomes a delta-function which confines the longitudinal component of the gauge field and its fluctuation to zero. This confines the whole analysis based on this ansatz to the subspace of transversal modes. Furthermore, under this constraint, there is no difference between the two sources N and η. This explains why the two approaches yield the same Green functions and the same Feynman rules in this limit.

This result also proves that the exponential factor in Eq. (17) does neither spoil the renormalizability of the theory nor affect its infrared properties. From a physical point of view this was to be expected. The phase does only involve the longitudinal component of the gauge field which does not affect the dynamics. Renormalization is in fact simpler in the manifestly gauge invariant approach. Since the symmetry is manifest at any step, all counter terms must necessarily be gauge invariant. Thus, for example, the wave function renormalization of any gauge field, Abelian or non-Abelian, must be the inverse of the corresponding coupling constant renormalization. One does not need to employ Ward identities to establish such relations.

4 The Standard Model

Gauge-invariant sources are readily constructed for all particles in the Standard Model of electroweak interactions. First we observe that the complex Higgs-doublet field ϕ and its conjugate field

$$\tilde{\phi} = i\tau^2\phi^* \tag{90}$$

provide a natural reference frame for $SU_L(2)$-valued fields [15]. In what follows we will furthermore use the decomposition

$$\phi = \frac{m}{\sqrt{\lambda}} RU \ , \tag{91}$$

where the unitary field U, satisfying $U^\dagger U = 1$, describes the three Goldstone bosons while the radial component R represents the Higgs boson.

For up and down type fermions one may consider the composite fields

$$\begin{array}{rclrcl} \phi^\dagger \Psi^i_L & = & \frac{m}{\sqrt{\lambda}} R d^i_L \ , & \tilde{\phi}^\dagger \Psi^i_L & = & \frac{m}{\sqrt{\lambda}} R u^i_L \ , \\ \phi^\dagger \Psi^i_L & = & \frac{m}{\sqrt{\lambda}} R e^i_L \ , & \tilde{\phi}^\dagger \Psi^i_L & = & \frac{m}{\sqrt{\lambda}} R \nu^i_L \ , \end{array} \tag{92}$$

where the interpolating fermion fields

$$\begin{array}{rclrcl} d^i_L & = & U^\dagger \Psi^i_L \ , & u^i_L & = & \tilde{U}^\dagger \Psi^i_L \ , \\ e^i_L & = & U^\dagger \Psi^i_L \ , & \nu^i_L & = & \tilde{U}^\dagger \Psi^i_L \ , \end{array} \tag{93}$$

are $SU_L(2)$-invariant and have the same quantum numbers as their right-handed counterparts. The quantities Ψ_L^i are the left-handed iso-doublet fields. For the massive gauge bosons one may consider the composite fields

$$\begin{array}{rclcl} V_\mu^1 &=& i\tilde{\phi}^\dagger D_\mu \phi + i\phi^\dagger D_\mu \tilde{\phi} &=& \frac{m2}{\lambda} R^2 \mathcal{W}_\mu^1 \\ V_\mu^2 &=& -\tilde{\phi}^\dagger D_\mu \phi + \phi^\dagger D_\mu \tilde{\phi} &=& \frac{m2}{\lambda} R^2 \mathcal{W}_\mu^2 \\ V_\mu^3 &=& i\tilde{\phi}^\dagger D_\mu \tilde{\phi} - i\phi^\dagger D_\mu \phi &=& \frac{m2}{\lambda} R^2 \mathcal{Z}_\mu \end{array} \tag{94}$$

where the gauge boson fields

$$\mathcal{W}_\mu^+ = \frac{i}{2}\left(\tilde{U}^\dagger (D_\mu U) - (D_\mu \tilde{U})^\dagger U\right) \tag{95}$$

$$\mathcal{W}_\mu^- = \frac{i}{2}\left(U^\dagger (D_\mu \tilde{U}) - (D_\mu U)^\dagger \tilde{U}\right) \tag{96}$$

$$\mathcal{Z}_\mu = i\left(\tilde{U}^\dagger (D_\mu \tilde{U}) - U^\dagger (D_\mu U)\right) \tag{97}$$

$$\mathcal{A}_\mu = B_\mu + s_W^2 \mathcal{Z}_\mu \tag{98}$$

$$\mathcal{W}_\mu^\pm = \frac{1}{2}(\mathcal{W}_\mu^1 \mp i\mathcal{W}_\mu^2) \tag{99}$$

are also $SU_L(2)$-invariant. The covariant derivative

$$D_\mu = \partial_\mu - \frac{i}{2}\tau^a W_\mu^a - \frac{i}{2} Y B_\mu \tag{100}$$

involves the $U_Y(1)$ gauge field B_μ and the $SU_L(2)$ triplet W_μ^a. Up to a constant factor the composite fields V_μ^i in Eq. (94) correspond to the currents of the custodial symmetry $SU_c(2)$. The gauge-boson fields in Eqs. (95-98) are long known from the effective field theory analysis of the symmetry-breaking sector of the Standard Model [17].

External sources can be coupled to the $SU_L(2)$-invariant composite fields in Eqs. (92) and (94). In order to make them invariant under the full group $SU_L(2) \times U_Y(1)$ one may introduce phase factors for all charged particles in the same fashion as in Eq. (17), involving the Abelian gauge field B_μ. One may choose

$$\begin{aligned} \mathcal{L}_{source} &= \frac{1}{2} h \phi^\dagger \phi + k_{\mu\nu} B^{\mu\nu} + J^{a\mu} V_\mu^a \\ &+ \bar{N}_L^i \phi^\dagger \Psi_L^i + \bar{\Psi}_L^i \phi N_L^i + \bar{M}_L^i \tilde{\phi}^\dagger \Psi_L^i + \bar{\Psi}_L^i \tilde{\phi} M_L^i + \bar{N}_R^i \psi_R^i + \bar{\psi}_R^i N_R^i \,, \end{aligned} \tag{101}$$

with external fields N_L^i, M_L^i, N_R^i, J_μ^a, $k_{\mu\nu}$ and h. One has, for example,

$$J_\mu(x) = j_\mu(x) e^{-T\int \mathrm{d}^d y G_0(x-y)\partial_\mu B^\mu(y)} \tag{102}$$

with

$$T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{103}$$

The sources in Eq. (101) define $SU_L(2) \times U_Y(1)$-invariant Green functions of the fields in Eqs. (93) and (95-98), of the Higgs field R and of the right-handed fermion fields ψ_R^i. Note that the vacuum in the spontaneously broken phase corresponds to the value $R = 1$. The occurrence of the Higgs-boson field in the sources of the left-handed fermions and of the gauge bosons will give rise to external line renormalizations, which are under control. The effect of the phase was thoroughly investigated in Section 2.

To evaluate these Green functions perturbatively one may employ the gauge-invariant technique of Section 2, which is not restricted to the Abelian case [10].

In order to analyse how Green's functions defined through the sources in Eq. (101) relate to the familiar gauge-dependent ones we define the Faddeev-Popov ansatz with the gauge-fixing condition

$$\mathcal{L}_{GF} = \frac{-1}{2\xi_B g'^2}\left(\partial_\mu B^\mu\right)^2 + \frac{-1}{2\xi_W g^2}\left(\partial^\mu W_\mu^a - i\xi_W \frac{mg^2}{\sqrt{\lambda}}\left(U_0^\dagger \tau^a \phi - \phi^\dagger \tau^a U_0\right)\right)^2 , \tag{104}$$

and the unitary-gauge version of the sources given above, i.e., with

$$U \to U_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{105}$$

set in the source terms. The phase factors involving the longitudinal component of the $U_Y(1)$ gauge field B_μ are also dropped. Apart from external line renormalizations due to the presence of the Higgs field R this defines gauge-dependent Green functions commonly used in the literature.

As far as the $U_Y(1)$ symmetry is concerned we have exactly the same situation as in the theory of QED. The non-Abelian part of the electroweak group is a different matter. The source terms of our Faddeev-Popov ansatz for the Standard Model were defined as the unitary-gauge limit of the corresponding $SU_L(2) \times U_Y(1)$ invariant source terms. Hence, we expect Green's functions of this Faddeev-Popov ansatz to be the same as those of the manifestly gauge-invariant framework if the limits $\xi_B \to 0$ (Landau gauge with respect to the group $U_Y(1)$) and $\xi_W \to \infty$ (unitary gauge with respect to the group $SU_L(2)$) are taken. It is indeed straightforward to verify this statement at tree-level. We did not go beyond that.

Since a gauge-dependent calculation in unitary gauge is plagued by the bad ultra-violet behaviour of the gauge-boson propagators it is not clear how this statement generalises to loop corrections. However, one should be very careful not to draw any wrong conclusion from this relation between gauge-invariant and gauge-dependent Green functions at tree-level. The bad ultra-violet behaviour is a specific problem of the gauge-dependent approach in unitary gauge. It does not occur in the manifestly gauge-invariant approach we described. We particularly emphasise that loop corrections in the gauge-invariant framework have a decent ultra-violet behaviour [9]. For large momenta k all propagators fall off as k^{-2}.

We conclude this section with some remarks about renormalization. Dimensional arguments show that the source terms in Eq. (101) are renormalizable. The phase does not affect this property as shown in Section 2. Green's functions of the composite operators in Eq. (92) and (94), however, are more singular at short distances than (gauge-dependent) Green functions of the fields $\phi, \Psi_L^i, B_\mu, \ldots$. Time ordering of these operators gives rise to ambiguities and corresponding Green functions are only unique up to contact terms. In order to make the theory finite, these contact terms of dimension four need to be added to the Lagrangian. A detailed discussion of this point is deferred to an explicit analysis of the Standard Model within this framework [18]. As for the Abelian Higgs model this is already discussed in Ref. [9].

5 Summary

This work provides a manifestly gauge-invariant approach to analyse gauge theories with charged particles. Its two main ingredients are (1) Green functions of gauge-invariant operators and (2) Feynman rules which do not involve any kind of gauge fixing.

In the first part of this article we thoroughly discussed the theory of

QED. We considered Green functions of the fermion field ψ and of the electromagnetic field strength $F_{\mu\nu}$ which are generated by the vacuum-to-vacuum transition amplitude

$$\langle 0_{\rm out}|0_{\rm in}\rangle_{N,\bar{N},k_{\mu\nu}} = \langle 0|T\left[e^{i\bar{N}\psi_{\rm op}+i\bar{\psi}_{\rm op}N+ik_{\mu\nu}F^{\mu\nu}_{\rm op}}\right]|0\rangle\ . \tag{106}$$

In order to generate gauge-invariant Green functions the external sources must not couple to the gauge degree of freedom. Hence, the external source of the fermion, N, must be gauge-variant, with the same transformation properties as the fermion field itself. Moreover, the vacuum-to-vacuum transition amplitude would not even be defined, if the external field N were a fixed function of space-time.

A very convenient source with the afore mentioned property is of the form

$$N(x) = \eta(x)e^{-i\int {\rm d}^{\rm d}yG_0(x-y)\partial_\mu A^\mu(y)}\ , \tag{107}$$

where η is a singlet under the gauge-group. The free massless propagator $G_0(z)$ is defined in Eq. (18). With this source the amplitude in Eq. (106) generates gauge-invariant Green functions of the fermion field and of the electromagnetic field strength which are obtained by taking derivatives with respect to the external fields η and $k_{\mu\nu}$.

The phase in Eq. (107) does only involve the longitudinal component of the gauge field which does not affect the dynamics of the theory. Hence it does not produce any vertex. Furthermore, it can neither harm renormalizability nor affect the infrared behaviour of the theory. It has only one effect, to ensure that the extenal field η does not couple to the gauge degree of freedom.

We showed how the path-integral representation of the vacuum-to-vacuum transition amplitude in Eq. (106) can be evaluated perturbatively without fixing the gauge. The technique involves only the physical degrees of freedom. The gauge degrees of freedom, i.e., the longitudinal component of the gauge field, the phase of the fermion field, as well as their quantum fluctuations drop out completely. As a result, the propagator of the gauge field is only defined on the subspace of transversal modes. The application of this manifestly gauge-invariant technique is necessary in order to avoid the problems discussed in the introduction.

A remarkable result is that the gauge-invariant Green functions defined through Eqs. (106) and (107) are exactly the same as the gauge-dependent Green functions defined through the Faddeev-Popov ansatz in Eq. (77) in

Landau gauge. Feynman rules in both approaches are also the same. This is readily understood. In the gauge-dependent framework the gauge-field propagator is defined on the full space, involving transversal and longitudinal components. In Landau gauge the longitudinal components are switched off and all quantities are confined to the physical subspace of transversal modes.

In the second part of this article we showed how the gauge-invariant approach can be extended to the Standard Model of electroweak interactions. The Higgs-doublet field ϕ and its conjugate field provide a frame of reference for $SU_L(2)$-valued quantities. Since the symmetry is spontaneously broken this property can be used to define $SU_L(2)$-invariant fields for fermions and bosons. For neutral particles these fields are already invariant under the full electroweak group $SU_L(2) \times U_Y(1)$. For charged particles, on the other hand, one may consider external sources similar to the one given in Eq. (107) involving the longitudinal component of the $U_Y(1)$ gauge field B_μ. This provides $SU_L(2) \times U_Y(1)$-invariant sources for all particles of the theory. The technique to evaluate the corresponding gauge-invariant Green functions without gauge fixing is the same as for the Abelian case. For details on the Standard Model, in particular for the comparison of the gauge-invariant approach to the Faddeev-Popov ansatz, the reader is refered to Section 4.

We have not discussed whether the gauge-invariant approach described in this article can also be applied to the theory of QCD. The open question is whether one can construct colour-singlet operators which create single quarks or gluons. To do so one might, for example, attempt to generalise the source term given in Eq. (107). However, since colour is confined it may as well turn out that all physically relevant information can be extracted from Green's functions of operators that create pairs of these particles.

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References

[1] W. Beenakker et. al. *Reports of the Working Group on Physics at LEP2* CERN Yellow Rep. CERN 96-01, eds. G. Altarelli, T. Sjöstrand, and F. Zwirner

[2] G. Degrassi, B.A. Kniehl, and A. Sirlin, *Phys. Rev.* **D48** (1993) 3963.

[3] J.M. Cornwall, *Phys. Rev.* **D26** (1982) 1453 and in *Proc. of the French-American Seminar on Theoretical Aspects of Quantum Chromodynamics,* ed. J.W. Dash, (Centre de Physique Theorique, Report No. CPT-81/P-1345, Marseille, 1982);

[4] A. Denner, S. Dittmaier, and G. Weiglein, in *Proc. of the Ringberg Workshop "Perspectives for electroweak interactions in* $e^+ e^-$ *collisions",* Ringberg Castle, February 5-8, 1995, ed. B.A. Kniehl

[5] M. J. Herrero and E. R. Morales, *Nucl. Phys.* **B437** (1995) 319; S. Dittmaier and C. Grosse-Knetter, *Nucl. Phys.* **B459** (1996) 497;

[6] D. Espriu and J. Matias, *Phys. Lett.* **B341** (1995) 332; J. Matias, Padua U. preprint (1996), DFPD-96-TH-18.

[7] J.M. Cornwall and J. Papavassiliou, *Phys. Rev.* **D40** (1989) 3474;

J. Papavassiliou, *Phys. Rev.* **D41** (1990) 3179;

G. Degrassi and A. Sirlin, *Phys. Rev.* **D46** (1992) 3104.

[8] L.F. Abbott, *Nucl. Phys.* **B185** (1981) 189; *Acta. Phys. Pol.* **B13** (1982) 33 and references therein.

[9] A. Nyffeler and A. Schenk, *Phys. Rev.* **D53** (1996) 1494.

[10] E. Corrigan and P. Goddard, *in* "Geometrical and Topological Methods in Gauge Theories" (J. P. Harnad and S. Shnider, Eds.), Proceedings, Montreal, 1979; Lecture Notes in Physics 129, Springer-Verlag, Berlin, 1980.

H. Leutwyler, *Nucl. Phys.* **B179** (1981) 129.

[11] Particle Data Group, Review of Particle Properties, *Phys. Rev.* **D54** (1996) 1.

[12] T. Appelquist and C. Bernard, *Phys. Rev.* **D22** (1980) 200;

A. C. Longhitano, *Phys. Rev.* **D22** (1980) 1166;

T. Appelquist, *in* "Gauge Theories and Experiments at High Energies" (K. C. Bowler and D. G. Sutherland, Eds.), Scottish University Summer School in Physics, St. Andrews, 1980;

A. C. Longhitano, *Nucl. Phys.* **B188** (1981) 118.

[13] A. Nyffeler and A. Schenk, *Ann. Phys.* **241** (1995) 301.

[14] L. D. Faddeev and V. N. Popov, *Phys. Lett.* **B25** (1967) 29.

[15] G. 't Hooft, *Acta Universitatis Wratislavensis* **368** (1976) 345.

[16] A. M. Polyakov, *Nucl. Phys.* **B120** (1977) 429.

[17] M. S. Chanowitz, M. Golden and H. Georgi, *Phys. Rev.* **D36** (1987) 1490

[18] A. Nyffeler and A. Schenk, in preparation.