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Two-Loop Corrections to the Electromagnetic Vertex for Energies close to Threshold

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Abstract

Two-loop contributions to the electromagnetic form factors are calculated in the kinematic regime close to the fermion-antifermion threshold. The results are presented in an expansion in the velocity β of the fermions in the c.m. frame up to next-to-next-to leading order in β. The existence of a new Coulomb singularity logarithmic in β, which is closely related to the $\mathcal{O}(\alpha^2 \ln \alpha)$ corrections known from positronium decays, is demonstrated. It is shown that due to this Coulomb singularity $\mathcal{O}(\alpha^2)$ relativistic corrections to the non-relativistic cross section of heavy fermion-antifermion pair production in e^+e^- annihilation cannot be determined by means of conventional multi-loop perturbation theory.

1 Introduction

In view of future experiments (NLC, B-factory, τ-charm factory) where heavy quark-antiquark pairs will be produced in the kinematic region close to the threshold and a large amount of data can be expected, it is a very attractive idea that an extraction of the strong coupling α_s at a specific scale (or equivalently $\Lambda_{\rm QCD}$) might be possible which is accurate enough to allow for a serious comparison to complementary determinations of α_s from high energy experiments, where quark masses are much smaller than the relevant energy scales. Such an analysis would be an extremely important test of QCD. In recent literature two attempts can be found [1, 2] where such an analysis has been carried out based on present data on properties of $b\bar{b}$ mesons and on theoretical calculations involving well known results in the non-relativistic limit. The results of these analyses are somewhat controversial indicating that a better understanding of the structure and size of relativistic corrections to the non-relativistic limit and of the interplay of these corrections with non-perturbative effects is mandatory.

The framework in which relativistic corrections can be determined systematically in a very elegant way is non-relativistic quantum chromodynamics (NRQCD) [3] which is based on the concept of effective field theories. NRQCD consists of a non-relativistic Schrödinger field theory with a Coulomb-like QCD potential whereby relativistic effects are incorporated by introduction of higher dimensional operators in accordance to the underlying symmetries. In order to render NRQCD equivalent to QCD the NRQCD Lagrangian has to be matched to predictions in the framework of conventional multi-loop perturbation theory. This procedure leads to, in general, divergent renormalization constants multiplying the operators in the NRQCD Lagrangian and is essentially equivalent to a separation of short- and long-distance effects. As far as the decay and production properties of a heavy quark-antiquark pair involving single photon annihilation in the threshold regime are concerned the relevant parts of the NRQCD Lagrangian have only been renormalized at leading and next-to-leading order in α_s so far [4].

In this letter we present the two-loop contributions to the electromagnetic vertex describing the decay of a virtual photon into two massive fermions in the kinematic regime where the squared photon four momentum is close to four times the squared fermion mass. The calculation is performed in the framework of QED where only one fermion species with mass M and electronic charge e exists. The result is presented up to next-to-next-to-leading order (NNLO) in an expansion in

$$\beta \,=\, \sqrt{1-4\,\frac{M^2}{q^2+i\epsilon}}\,, \tag{1}$$

which is equal to the velocity of the fermions in the c.m. frame above threshold[1], $\sqrt{q^2}$ being the c.m. energy. We analyse the structure and form of the results and demonstrate the existence of a new logarithmic Coulomb singularity occurring at NNLO in the velocity expansion. In particular, we will study the impact of this singularity on the massive fermion-antifermion pair production cross section slightly above the threshold. In the framework of QCD our two-loop results represent all two-loop contributions involving the color factor C_F^2 (from exchange of two virtual gluons) and C_FT (from the exchange of one gluon with the insertion of the fermion-antifermion vacuum polarization) and, therefore, are a gauge invariant subset of all two-loop QCD contributions in the threshold regime[2].

[1]Thus β will be called "velocity" for the rest of this paper. In this paper we use the notion "leading order" (and NLO, NNLO, NNNLO) exclusively for the expansion in the velocity.

[2]The two-loop contributions arising from the virtual effects of massless fermions have been calculated in [5] for all

The two-loop contributions calculated in this work represent a first step toward a two-loop renormalization of the NRQCD Lagrangian describing single photon annihilation processes involving heavy quark-antiquark pairs. In particular, they are a crucial input for the determination of NNLO relativistic corrections for the single photon annihilation contributions to decay and production of heavy quark-antiquark bound states and for the production of heavy quark-antiquark pairs in e^+e^- collisions slightly above threshold. In the framework of QED the result is essential for the determination of the single photon annihilation contributions to the $\mathcal{O}(\alpha^6)$ triplet-singlet hyperfine splitting of the positronium ground state.

The content of this work is organized as follows: in Section 2 we explain the notation and introduce the electromagnetic form factors relevant for our calculations and discussions. In Section 3 we reanalyse the well-known one-loop contributions to the form factors in the threshold region. We discuss the structure and properties of the individual coefficients of the expansion in small β and derive predictions for the form of the two-loop corrections based on the factorization of long- and short-distance contributions. In Section 4 the two-loop corrections are explicitly calculated using the dispersion integration technique. It is demonstrated that the predictions of Section 3 are realized and the logarithmic Coulomb singularity is discussed. Section 5 contains a summary.

2 Notation and Definition of the Electromagnetic Form Factors

It is common to parameterize radiative (multi-loop) corrections to the electromagnetic vertex, describing the decay of a photon with virtuality q^2 into a fermion-antifermion pair, in terms of the Dirac (F_1) and the Pauli (F_2) form factors. They are defined through the relation

$$\bar{u}(p')\,\Lambda_\mu^{em}\,v(p) \,=\, i\,e\,\bar{u}(p')\left[\gamma_\mu\,F_1(q^2) + \frac{i}{2\,M}\,\sigma_{\mu\nu}\,q^\nu\,F_2(q^2)\right]v(p)\,, \tag{2}$$

where

$$q \,=\, p + p' \quad \text{and} \quad \sigma_{\mu\nu} \,=\, \frac{i}{2}\,[\,\gamma_\mu,\gamma_\nu\,]\,.$$

Expanded in the number of loops, which corresponds to an expansion in powers of the fine structure constant α, the form factors F_1 and F_2 read

$$\begin{aligned} F_1(q^2) &= 1 + \left(\frac{\alpha}{\pi}\right) F_1^{(1)}(q^2) + \left(\frac{\alpha}{\pi}\right)^2 F_1^{(2)}(q^2) + \cdots\,, \\ F_2(q^2) &= \left(\frac{\alpha}{\pi}\right) F_2^{(1)}(q^2) + \left(\frac{\alpha}{\pi}\right)^2 F_2^{(2)}(q^2) + \cdots\,. \end{aligned} \tag{3}$$

The use of F_1 and F_2 is particularly convenient for the kinematic point $q^2 = 0$ because $F_2(0) = (g_f - 2)/2$ is directly related to the gyro-magnetic ratio of the fermion and because $F_1(0) = 1$ (i.e. $F_1^{(n)}(0) = 0$ for $n = 1, 2, \ldots, \infty$) due to gauge invariance. These properties are useful if dispersion relation techniques are used to calculate higher loop contributions because overall UV divergences to $F_1^{(n)}$ ($n = 1, 2, \ldots, \infty$) can be automatically renormalized by using once-subtracted dispersion relations. For F_2, on the other hand, no overall UV divergences exist which makes the use of unsubtracted

ratios M^2/q^2 above threshold and will not be discussed in this work.

dispersion relations convenient. Since the determination of our two-loop results relies on the dispersion relation technique we will use the form factors F_1 and F_2 for the actual calculations.

For physical applications in the threshold region, where $q^2 \approx 4M^2$, however, the use of the combinations

$$G_m = F_1 + F_2 \,, \tag{4}$$

$$G_e = F_1 + \frac{s}{4\,M^2}\,F_2 \,. \tag{5}$$

is more appropriate. This can be easily seen by considering the contributions of the form factors F_1 and F_2 to the cross section for the production of a fermion-antifermion pair (with fermion mass M) in e^+e^- annihilation. Taking the colliding electrons and positrons as massless one arrives at the following angular distribution for the produced fermion pairs for the c.m. energy $\sqrt{q^2}$ above threshold

$$\frac{\mathrm{d}\,\sigma(\mathrm{e^+e^-} \to \mathrm{f\bar{f}})}{\mathrm{d}\,\Omega} = \frac{\alpha^2\,\beta}{4\,q^2}\left[\,|G_m|^2\,(1+\cos^2\theta) + \frac{4\,M^2}{q^2}|G_e|^2\,\sin^2\theta\,\right], \tag{6}$$

where θ is the deflection angle. The corresponding expression for the total cross section reads ($\sigma_{pt} = 4\pi\alpha^2/3q^2$)

$$R \equiv \frac{\sigma(e^+e^- \to f\bar{f})}{\sigma_{pt}} = \beta\left[\,|G_m|^2 + \frac{1}{2}\,(1-\beta^2)\,|G_e|^2\,\right]. \tag{7}$$

G_m and G_e are called magnetic and electric form factors, respectively [6]. They can be easily identified as the total spin projection (relative to the electron direction) ± 1 and 0 amplitudes describing the produced fermion-antifermion pair in a triplet ($J^{PC} = 1^{--}$) state. Because the fermion-antifermion production cross section represents one of the most important applications of the corrections to the electromagnetic vertex we will discuss the structure and properties of the corrections by analysing the moduli squared of the magnetic and electric form factors above threshold. Their expansion in the number of loops (i.e. in powers of the fine structure constant) reads

$$|G_m|^2 = 1 + \left(\frac{\alpha}{\pi}\right) g_m^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 g_m^{(2)} + \cdots \,,$$

$$|G_e|^2 = 1 + \left(\frac{\alpha}{\pi}\right) g_e^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 g_e^{(2)} + \cdots \,. \tag{8}$$

We finally would like to emphasize that throughout this paper the fermions are understood as stable particles and that the on-shell renormalization scheme is employed, where $\alpha = 1/137$ and M is the fermion pole mass.

3 One-Loop Results

Analytic expressions for the one-loop contributions to the electromagnetic vertex valid for all energies are well known since quite a long time [7, 8]. In this section we reanalyse the one-loop contributions in the threshold region in the velocity expansion as a preparation for the examination of the two-loop contributions in Section 4.

Regularizing the soft photon infrared divergences with a fictitious small photon mass λ, where the hierarchy $\lambda/M \ll |\beta| \ll 1$ is understood, the one-loop contributions to the electromagnetic form factors F_1 and F_2 assume the form

$$F_1^{(1)}(q^2) \stackrel{\beta\to 0}{=} i\,\frac{\pi}{2\,\beta}\left[\,\ln\Big(-\frac{2\,i\,\beta\,M}{\lambda}\Big)-\frac{1}{2}\right]-\frac{3}{2}+i\,\frac{\pi\,\beta}{2}\left[\,\ln\Big(-\frac{2\,i\,\beta\,M}{\lambda}\Big)-\frac{1}{2}\right] \\ -\frac{4}{3}\left[\,\ln\Big(\frac{M}{\lambda}\Big)+\frac{5}{24}\right]\beta^2+\mathcal{O}(\beta^3)\,, \tag{9}$$

$$F_2^{(1)}(q^2) \stackrel{\beta\to 0}{=} i\,\frac{\pi}{4\,\beta}-\frac{1}{2}-i\,\frac{\pi\,\beta}{4}+\frac{1}{3}\beta^2+\mathcal{O}(\beta^3) \tag{10}$$

in the velocity expansion up to NNNLO. Expressions (9) and (10) are valid above as well as below the threshold point, $q^2 = 4M^2$, and lead to the following one-loop contributions to the moduli squared of the magnetic and electric form factors above the threshold

$$\Big(\frac{\alpha}{\pi}\Big)\,g_m^{(1)}(q^2) \stackrel{\beta\to 0}{=} \frac{\alpha\,\pi}{2\,\beta}-4\,\frac{\alpha}{\pi}+\frac{\alpha\,\pi\,\beta}{2}-\frac{\alpha}{3\,\pi}\left[\,8\,\ln\Big(\frac{M}{\lambda}\Big)-\frac{1}{3}\right]\beta^2+\mathcal{O}(\beta^3)\,, \tag{11}$$

$$\Big(\frac{\alpha}{\pi}\Big)\,g_e^{(1)}(q^2) \stackrel{\beta\to 0}{=} \frac{\alpha\,\pi}{2\,\beta}-4\,\frac{\alpha}{\pi}+\frac{\alpha\,\pi\,\beta}{2}-\frac{8\,\alpha}{3\,\pi}\left[\,\ln\Big(\frac{M}{\lambda}\Big)+\frac{1}{3}\right]\beta^2+\mathcal{O}(\beta^3)\,. \tag{12}$$

For the rest of this section we will discuss the individual terms in the velocity expansion displayed in eqs. (11) and (12). We would like to emphasize that most of the issues which are mentioned are well known and have been noted before at various places throughout the literature. However, we think that a review of these topics is necessary for a better understanding of the structure of the two-loop results presented in Section 4 and the new information contained in them.

Expressions (11) and (12) exhibit the well known soft photon divergence $\propto \ln(M/\lambda)$ which arises from the masslessness of the photon. This divergence occurs at order β^2 and would cancel with the corresponding soft photon divergence coming from the process of real radiation of one photon off one of the fermions according to the Kinoshita-Lee-Nauenberg theorem [9, 10]. The fact that the divergent term $\ln(M/\lambda)$ is suppressed by β^3 relative to the leading contribution in the expansion in β is expected at any loop level because close to threshold the real radiation of one photon results in an additional factor β from the phase space needed for the photon and a factor β^2 from the square of the dipole matrix element[3]. Because the soft photon $\ln(M/\lambda)$ divergence indicates the inadequacy of a pure fermion-antifermion final state and the need for the introduction of a higher fock fermion-antifermion-photon state, the β^3 suppression allows us to conclude that the notion of a pure fermion-antifermion state is consistent if we are only interested in NNLO accuracy in the expansion in β.

The leading term in the velocity expansion in eqs. (11) and (12) is the well known Coulomb singularity which diverges for $\beta \to 0$. Similar to the soft photon divergence discussed above the Coulomb singularity arises from the fact that the photon is massless and represents a long-distance effect. The Coulomb singularity, however, is of completely different nature. Whereas the soft photon singularity indicates the inadequacy of a pure fermion-antifermion state beyond NNLO in the velocity expansion

[3] It should be noted that this statement is equivalent to the fact that contributions from the non-instantaneous (i.e. transverse) exchange of photons among the fermion-antifermion pair are suppressed by β^3 with respect to the leading contributions in the velocity expansion. As an example, this feature is apparent in a $^3S_1$, $J^{PC} = 1^{--}$ fermion-antifermion bound state, where the velocity β of the fermions is of order α. There, the exchange of non-instantaneous photons leads to the Lamb shift which represents a $\mathcal{O}(\alpha^3)$ correction relative to the Coulomb energy levels. (See also [11].)

the Coulomb singularity reveals that in the non-relativistic limit (corresponding to the leading order in the velocity expansion) the photon-mediated interaction between the fermion-antifermion pair cannot be described in an expansion in Feynman diagrams, where a diagram with a larger number of loops (corresponding to a larger number of exchanged photons) would represent a higher order correction. Rather, a resummation of diagrams with any number of exchanged photons is needed to arrive at a sensible description of the interaction between the fermion-antifermion pair. The leading contribution in the velocity expansion is obtained by resummation of diagrams with instantaneous Coulomb exchanges of longitudinal photons (in the Coulomb gauge). This procedure can be explicitly carried out by calculating the normalized wave function at the origin, $\Psi_E(0)$, to the Schrödinger equation describing a non-relativistic fermion-antifermion pair with a Coulomb interaction potential for positive energies $E = M\beta^2$. The result of this calculation reads (see e.g. [8, 12, 13])

$$|G_m|^2_{\rm LO} \,=\, |G_e|^2_{\rm LO} \,=\, |\Psi_{M\beta^2}(0)|^2 \,=\, \frac{z}{1-\exp{(-z)}}\,, \tag{13}$$

where

$$z \,\equiv\, \frac{\alpha\,\pi}{\beta}\,, \tag{14}$$

and is often called "Sommerfeld factor" in the literature. The $1/\beta$ Coulomb singularity in eqs. (11) and (12) can be recovered as the $\mathcal{O}(\alpha)$ contribution in the expansion of the Sommerfeld factor for $\alpha \ll \beta$,

$$\frac{z}{1-\exp{(-z)}} \stackrel{\alpha\ll\beta}{=} 1+\frac{z}{2}+\frac{z^2}{12}+\mathcal{O}(\alpha^3)\,. \tag{15}$$

This, on the other hand, also shows that the velocity expansion of the perturbative (in the number of loops) series can only be applied in the limit $\alpha \ll \beta \ll 1$, where an expansion in the number of loops (i.e. in α) is justified[4]. It is worth to study the effect of this resummation: inserting the Sommerfeld factor into the formula for the cross section, eq. (7), we get at threshold

$$R \,\sim\, \frac{3}{2}\,\beta\,\frac{z}{1-\exp{(-z)}} \,\stackrel{\beta\to 0}{\longrightarrow}\, \frac{3}{2}\,\alpha\,\pi\,, \tag{16}$$

which is the correct result according to non-relativistic quantum mechanics. On the other hand, if we naively use the one-loop result (i.e. expansion in small α), we obtain

$$R \,\sim\, \frac{3}{2}\,\beta\,(1+\frac{z}{2}) \,\stackrel{\beta\to 0}{\longrightarrow}\, \frac{3}{4}\,\alpha\,\pi\,. \tag{17}$$

Clearly, the perturbative calculation in the number of loops, which is based on the assumption that α is a valid expansion parameter close to threshold, gives a prediction for R at threshold which deviates from the correct one by a factor of one half.

The next-to leading contribution in the velocity expansion in eqs. (11) and (12), $-4\,\alpha/\pi$, represents a short-distance correction and can be understood as a finite $\mathcal{O}(\alpha)$ renormalization of

[4]It should be noted that the region of convergence of the Taylor expansion

$$\frac{z}{1-\exp{(-z)}} \,=\, 1+\frac{z}{2}+\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{B_n\,z^{2n}}{(2n)!}\,,$$

where B_n are the Bernoulli numbers ($B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42\,,\ldots$), is $|z| < 2\pi \,\Leftrightarrow\, |\beta| > \alpha/2$. This shows that for phenomenological applications a resummation of the leading order contributions in the velocity expansion to any number of loops is mandatory in the kinematic regime $|\beta| \lesssim \alpha$.

the electromagnetic current which produces the fermion-antifermion pair in the threshold region. The short-distance character of this $\mathcal{O}(\alpha)$ correction has been demonstrated explicitly by the calculation of the BLM (Brodsky-Lepage-Mackenzie [14]) scale in the coupling governing the $-4\,\alpha/\pi$ contribution [5, 15, 1]. This BLM scale is of order the fermion mass M and indicates that the $-4\,\alpha/\pi$ contribution represents a correction to the fermion-antifermion production process which occurs at short distances of order $1/M$. In contrast, the BLM scale of the coupling in the leading term in the velocity expansion, $\alpha\pi/2\beta$, is of order of the relative momentum of the fermion-antifermion pair, $M\beta$ [5, 15], indicating that the latter contribution belongs to the fermion-antifermion wave function. As a consequence the leading order (long-distance) contributions contained in the Sommerfeld factor and the short-distance corrections are expected to factorize which leads to

$$|G_m|^2_{\rm NLO} \;=\; |G_e|^2_{\rm NLO} \;=\; \frac{z}{1-\exp{(-z)}}\,\bigg(1-4\,\frac{\alpha}{\pi}\bigg) \tag{18}$$

for the NLO expressions in the velocity expansion of the moduli squared of the magnetic and electric form factors in the threshold region. It should be noted that the factorized result (18) resums all contributions $(\alpha/\beta)^n\times[1,\alpha]$, $n=0,1,2,\ldots,\infty$. Because no $(\alpha/\beta)^n\beta$ contributions exist[5], expression (18) unambiguously predicts the leading and next-to-leading order contributions in the velocity expansion for all $g^{(n)}_{m/e}$, $n=2,3,\ldots,\infty$.

The NNLO term in the velocity expansion in eqs. (11) and (12), $\alpha\pi\beta/2$, has not received much attention in the literature so far. Its structure, which involved the same power of π and the same coefficient $1/2$ as the LO term in the velocity expansion, strongly implies that it is of long-distance origin and therefore belongs to the Sommerfeld factor. This is in accordance to the observation that the BLM scale in the coupling of the term $\alpha\pi\beta/2$ is of order $M\beta$ rather than M [5]. The relativistic extension of the Sommerfeld factor (including $\mathcal{O}(\beta^2)$ corrections) should then read

$$\frac{\tilde z}{1-\exp(-\tilde z)}\,, \qquad \tilde z\equiv\frac{\alpha\,\pi}{\beta}\,(1+\beta^2)\,. \tag{19}$$

Although the arguments given above in favor of expression (19) are far from being a strict proof the form of $\tilde z$ is very convincing because it indicates that the relativistic relative velocity v_{rel} of the fermion-antifermion pair in the c.m. frame is involved in the argument of the Sommerfeld factor if $\mathcal{O}(\beta^2)$ relativistic corrections are taken into account,

$$\tilde z \;=\; \frac{2\,\alpha\,\pi}{v_{rel}}\,, \qquad v_{rel} \;=\; \frac{2\,\beta}{1+\beta^2}\,. \tag{20}$$

Combining expression (19) with the short-distance factor $(1-4\alpha/\pi)$ and taking into account that no soft photon divergence $\propto \ln(M/\lambda)$ arises up to NNLO in the velocity expansion we can now predict that the two-loop contributions to $|G_{m/e}|^2$ must have the form

$$g^{(2)}_{m/e}(q^2) \;\stackrel{\beta\to 0}{=}\; \frac{\pi^4}{12\,\beta^2}\;-\;2\,\frac{\pi^2}{\beta}\;+\;\frac{\pi^4}{6}\;+\;\bigg[\,\mbox{finite terms without }\pi^4\bigg]\;+\;\mathcal{O}(\beta)\,. \tag{21}$$

We want to emphasize the the $\mathcal{O}(1/\beta^2)$, $\mathcal{O}(1/\beta)$ and $\mathcal{O}(\beta^0\pi^4)$ contributions on the r.h.s. of eq. (21) are an unambiguous prediction and have to be recovered in the explicit two-loop result if the concept

[5] Pure β-dependent corrections to the Sommerfeld factor are of kinematic origin and therefore expected to be of NNLO in the velocity expansion, i.e. $\propto(\alpha/\beta)^n\beta^2$, $n=0,1,2,\ldots,\infty$.

of factorization in the threshold regime is valid. It should be noted that up to NNLO in the velocity expansion only the $\mathcal{O}(\beta^0)$ contributions symbolized by [finite terms without π^4] contain new two-loop information.

4 Two-Loop Results

To determine the two-loop contributions to the electromagnetic form factors F_1 and F_2 in the velocity expansion we use the dispersion integration technique. For that we have to integrate over the absorptive parts $\mathrm{Im}F_{1/2}^{(2)}$ which have been determined a long time ago by Barbieri, Mignaco and Remiddi [16],

$$
\begin{aligned}
F_1^{(2)}(q^2) \;&=\; -\frac{4\,M^2\,q^2}{q^2-4\,M^2}\,F_1^{\prime\,(2)}(0) \\
&\quad +\frac{1}{\pi}\,\frac{q^4}{q^2-4\,M^2}\int\limits_{4\,M^2}^{\infty}\frac{dq'^{\,2}}{q'^{\,2}(q'^{\,2}-q^2-i\epsilon)}\,\frac{q'^{\,2}-4\,M^2}{q'^{\,2}}\,\mathrm{Im}F_1^{(2)}(q'^{\,2})\,,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
F_2^{(2)}(q^2) \;&=\; -\frac{4\,M^2}{q^2-4\,M^2}\,F_2^{(2)}(0) \\
&\quad +\frac{1}{\pi}\,\frac{q^2}{q^2-4\,M^2}\int\limits_{4\,M^2}^{\infty}\frac{dq'^{\,2}}{q'^{\,2}-q^2-i\epsilon}\,\frac{q'^{\,2}-4\,M^2}{q'^{\,2}}\,\mathrm{Im}F_2^{(2)}(q'^{\,2})\,.
\end{aligned}
\tag{23}
$$

We would like to mention that relations (22) and (23) are equivalent to the common once-subtracted and unsubtracted dispersion relations. We use (22) and (23) because they do not run into non-analyticity problems in the integration region where $q'^2 - 4M^2$ is of order λ^2 if the limit $\lambda \to 0$ is already taken before the integration. Since the absorptive parts in [16] are given in exactly this limit (22) and (23) are more convenient because in them the integration regime $q'^2 - 4M^2 \approx \lambda^2$ is strongly suppressed. The (low) price one has to pay is that the $\mathcal{O}(\alpha^2)$ fermion charge radius [17, 18],

$$
F_1^{\prime\,(2)}(0) \,=\, \frac{1}{M^2}\left[\,\frac{\pi^2}{6}\left(3\,\ln 2-\frac{49}{72}\right)-\frac{3}{4}\,\zeta_3-\frac{4819}{5184}\,\right],
\tag{24}
$$

and the $\mathcal{O}(\alpha^2)$ anomalous magnetic moment [19, 20],

$$
F_2^{(2)}(0) \,=\, \frac{\pi^2}{12}\left(-\,6\,\ln 2+1\right)+\frac{3}{4}\,\zeta_3+\frac{197}{144}\,,
\tag{25}
$$

have to be taken as an input[6]. Details for the quite lengthy but straightforward calculation of the integrals (22) and (23), which requires strong support of algebraic manipulation programs, shall be presented elsewhere.

[6]This fact has already been pointed out in [16]. We also refer the reader to this reference for a more thorough discussion of the problems which occur in the integration region $q'^2 - 4M^2 \approx \lambda^2$.

The final results for the two-loop contributions to F_1 and F_2 up to NNLO in the velocity expansion read

$$F_{1,2\gamma}^{(2)} \stackrel{\beta\to 0}{=} -\frac{\pi^2}{8\,\beta^2}\left[\frac{\pi^2}{6}+\left(\ell^2-\ell+\frac{1}{3}\right)\right]+i\,\frac{\pi}{4\,\beta}\left[-3\,\ell+1\right] \qquad (26)$$
$$-\left[\frac{\pi^4}{24}+\frac{\pi^2}{4}\left(\ell^2-\ell+\frac{23}{15}\ln(-i\,\beta)+\frac{7}{10}\ln 2+\frac{73}{50}\right)+\frac{9}{80}\left(9\,\zeta_3-\frac{421}{27}\right)\right]+\mathcal{O}(\beta)\,,$$

$$F_{1,f}^{(2)} \stackrel{\beta\to 0}{=} -\frac{13\,\pi^2}{45}+\frac{37}{12}+\mathcal{O}(\beta^2)\,, \qquad (27)$$

$$F_{2,2\gamma}^{(2)} \stackrel{\beta\to 0}{=} -\frac{\pi^2}{8\,\beta^2}\left[\ell-\frac{1}{3}\right]-i\,\frac{\pi}{4\,\beta}\left[\ell+1\right]$$
$$+\left[\frac{\pi^2}{20}\left(\ln(-i\,\beta)+\frac{101}{6}\ln 2-\frac{559}{45}\right)+\frac{1}{80}\left(41\,\zeta_3+\frac{269}{3}\right)\right]+\mathcal{O}(\beta)\,, \qquad (28)$$

$$F_{2,f}^{(2)} \stackrel{\beta\to 0}{=} \frac{\pi^2}{15}-\frac{23}{36}+\mathcal{O}(\beta^2)\,, \qquad (29)$$

where

$$\ell \equiv \ln\left(-\frac{2\,i\,\beta\,M}{\lambda}\right) \qquad (30)$$

and, as in the one-loop case, the hierarchy $\lambda/M \ll |\beta| \ll 1$ is understood. In eqs. (26)–(29) the contributions from diagrams with two photons (subscript 2γ) and from the diagrams with one photon and the insertion of the fermion-antifermion vacuum polarization[7] (subscript f) are displayed separately. This will facilitate the application in the framework of QCD where both types of contributions are multiplied by the different color factors C_F^2 and $C_F T$, respectively, and represent gauge invariant subsets of the full QCD two-loop contributions.

The results (26)–(29) lead to the following two-loop contributions to the moduli squared of the magnetic and electric form factors above threshold up to NNLO in the velocity expansion

$$g_m^{(2)}(q^2) \stackrel{\beta\to 0}{=} \frac{\pi^4}{12\,\beta^2}-2\,\frac{\pi^2}{\beta}+\frac{\pi^4}{6}+\pi^2\left(-\frac{2}{3}\ln\beta+\frac{4}{3}\ln 2-\frac{29}{12}\right)-\zeta_3+\frac{527}{36}+\mathcal{O}(\beta)\,, \qquad (31)$$

$$g_e^{(2)}(q^2) \stackrel{\beta\to 0}{=} \frac{\pi^4}{12\,\beta^2}-2\,\frac{\pi^2}{\beta}+\frac{\pi^4}{6}+\pi^2\left(-\frac{2}{3}\ln\beta+\frac{4}{3}\ln 2-\frac{7}{3}\right)-\zeta_3+\frac{527}{36}+\mathcal{O}(\beta)\,. \qquad (32)$$

It is evident that the prediction made in the previous section based on the one-loop corrections and on the factorization of long- and short-distance contributions (see eq. (21)) are indeed realized by our explicit two-loop result confirming the statements given in Section 3. As a consequence only the $\mathcal{O}(\beta^0)$ terms in eqs. (31) and (32) essentially contain new information.

The most conspicuous feature of the $\mathcal{O}(\beta^0)$ contributions in eqs. (31) and (32) is the term $\ln(\beta)$. Similar to the $1/\beta^2$ Coulomb singularity exhibited in the leading term in the velocity expansion, it indicates the breakdown of the conventional perturbation series in the number of loops in the limit $\beta\to 0$. The existence of this logarithm can be understood from the fact that two scales are involved in the kinematic regime near threshold, the fermion mass M and the three momentum of the fermion and antifermion in the c.m. frame $p\equiv M\beta$. The logarithm of the velocity β is therefore actually the logarithm of the ratio of these two scales, $\ln(p/M)$. Because the soft scale p is characteristic

[7] The two-loop corrections $F_{1,f}^{(2)}$ and $F_{2,f}^{(2)}$ have already been calculated before in [5] for all energies above threshold.

for the fermion-antifermion wave function and not relevant for the production mechanism of the fermion-antifermion pair (which involves only the hard scale M), the $\alpha^2\ln(p/M)$ term in eqs. (31) and (32) should occur with the same coefficient in the $\mathcal{O}(\alpha^2)$ corrections to the positronium decay rates. For a viable comparison, however, we also have to include the fermion-antifermion vacuum polarization effects coming from the fact that the fermion-antifermion pair, which is in a $J^{PC}=1^{--}$ state, can virtually annihilate into one photon. This can be easily achieved by multiplying $|G_{m/e}|^2$ by the factor $|1+\Pi|^{-2}$, where Π is the one-particle-irreducible vacuum polarization function. The two-loop contribution to Π also contains a logarithm of β in the velocity expansion [21]. This leads to the additional contribution $\alpha^2\ln(\beta)$ which has to added to $-2\,\alpha^2\ln(\beta)/3$ from $|G_{m/e}|^2$. (Actually the spin average of the logarithmic terms in expressions (31) and (32) has to be taken. This trivially results in $-2\,\alpha^2\ln(\beta)/3$ because the logarithmic term is universal in both spin amplitudes.) Because the relative momentum of the electron-positron pair in the positronium is of order $M\alpha$ we can expect that the $\mathcal{O}(\alpha^2)$ corrections to the $({}^3S_1,\ J^{PC}=1^{--})$ orthopositronium decay rate should contain the contribution $\alpha^2\ln(\alpha)/3$. This logarithmic $\mathcal{O}(\alpha^2)$ correction has indeed been found by explicit calculations of higher order correction to the orthopositronium decay rate [22]. We therefore have to conclude that the $\ln(\beta)$ term in eqs. (31) and (32) represents a new type of Coulomb singularity which, similar to the power-like $1/\beta^n$ singularities, requires a resummation of contributions to all orders in the number of loops[8]. How such a resummation has to be carried out for the $\ln(\beta)$ term in the vacuum polarization has been demonstrated in [21].

Finally, we want to discuss the impact of the $\ln(\beta)$ singularity on the cross section of fermion-antifermion production very close to threshold, see eqs. (6) and (7). Because the moduli squared of the magnetic and electric form factors are multiplied by the phase space factor β one might naively think that the $\ln(\beta)$ singularity is suppressed by β and does not affect the cross section for $\beta\to 0$. At this point we have to emphasize that the same would then be true for the short-distance correction, $-4\,\alpha/\pi$, in the one-loop contribution to $|G_{m/e}|^2$ because the latter also represents a $\mathcal{O}(\beta^0)$ term in the velocity expansion (see eqs. (11) and (12)). However, the one-loop short-distance correction survives for $\beta\to 0$, see eqs. (16) and (18). The resolution of this apparent contradiction comes from the fact that due to factorization (see eq. (18)) the one-loop short-distance correction is also contained in the $\mathcal{O}(1/\beta)$ term of the two-loop contribution to $|G_{m/e}|^2$ where it multiplies the $\mathcal{O}(\alpha)$ contribution of the expansion of the Sommerfeld factor for small α. This contribution does not vanish in the cross section for $\beta\to 0$ and illustrates the mechanism why the one-loop short-distance correction survives in this limit. In order to see that something similar happens to the $\ln(\beta)$ singularity in the two-loop results (31) and (32) let us have a closer look on the structure of the one- and two-loop contributions to the form factors F_1 and F_2. It has been shown by Yennie, Frautschi and Suura [24] that the infrared soft photon divergences exponentiate completely. Because real soft photon divergences in $|G_{m/e}|^2$ occur only beyond NNLO in the velocity expansion (see Section 3) all soft photon divergences which arise up to NNLO in the velocity expansion in eqs. (26)–(29) can be factorized into a divergent phase factor which is known as the *Coulomb phase*. In the moduli squared of the form factors this phase drops out. Since the Coulomb phase has to be considered as an intrinsic property of the fermion-

[8]At this point we would like to mention that the logarithmic Coulomb singularity has also been disussed in [23] in the framework of quarkonia decays. However, it is argued in [23] (and also in [4]) that this singularity (called "logarithmic infrared divergence" in [4]) would indicate that perturbative QCD could not be applied in the kinematic regime close to the threshold. We disagree with this conclusion, because we think that this singularity can be treated by a proper resummation of contributions to all orders in the number of loops.

antifermion wave function, where the relative momentum $2M\beta$ is a relevant scale, we can expect that the divergent phase factor should involve the logarithm of the ratio $2M\beta/\lambda$. This feature is indeed realized because the sum of Born, one-loop and two-loop contributions to F_1 and F_2 above threshold can be rewritten as

$$\begin{aligned}
& 1 + \Big(\frac{\alpha}{\pi}\Big)\, F_1^{(1)} + \Big(\frac{\alpha}{\pi}\Big)^2 \Big[F_{1,2\gamma}^{(2)} + F_{1,f}^{(2)} \Big] \\
& \quad \longrightarrow \; \exp\Big\{ i\,\frac{\alpha}{2}\,\Big(\frac{1}{\beta}+\beta\Big)\,\ell \Big\} \Big\{ 1 - \Big(\frac{\alpha}{\pi}\Big) \Big[i\,\frac{\pi}{4}\,\Big(\frac{1}{\beta}+\beta\Big) + \frac{3}{2} \Big] + \Big(\frac{\alpha}{\pi}\Big)^2 \Big[-\frac{\pi^2}{24\,\beta^2}\,\Big(\frac{\pi^2}{2}+1\Big) \\
& \qquad\qquad + i\,\frac{\pi}{4\,\beta} - \frac{\pi^4}{24} - \frac{\pi^2}{20}\,\Big(\frac{23}{3}\,\ln(-i\,\beta) + \frac{7}{2}\,\ln 2 + \frac{1177}{90}\Big) - \frac{9}{80}\,\Big(9\,\zeta_3 - 43\Big) \Big] \Big\}\,, \qquad (33)
\end{aligned}$$

$$\begin{aligned}
& \Big(\frac{\alpha}{\pi}\Big)\, F_2^{(1)} + \Big(\frac{\alpha}{\pi}\Big)^2 \Big[F_{2,2\gamma}^{(2)} + F_{2,f}^{(2)} \Big] \\
& \quad \longrightarrow \; \exp\Big\{ i\,\frac{\alpha}{2}\,\Big(\frac{1}{\beta}+\beta\Big)\,\ell \Big\} \Big\{ \Big(\frac{\alpha}{\pi}\Big) \Big[i\,\frac{\pi}{4}\,\Big(\frac{1}{\beta}-\beta\Big) - \frac{1}{2} \Big] + \Big(\frac{\alpha}{\pi}\Big)^2 \Big[\frac{\pi^2}{24\,\beta^2} \\
& \qquad\qquad - i\,\frac{\pi}{4\,\beta} + \frac{\pi^2}{20}\,\Big(\ln(-i\,\beta) + \frac{101}{6}\,\ln 2 - \frac{499}{45}\Big) + \frac{1}{80}\,\Big(41\,\zeta_3 + \frac{347}{9}\Big) \Big] \Big\}\,. \qquad (34)
\end{aligned}$$

The factorized expressions (33) and (34) predict that at the three-loop level the real parts of the form factors F_1 and F_2 contain the logarithmic and λ-independent $\mathcal{O}(1/\beta)$ contributions $-23\,\alpha^3\pi\ln(\beta)/240\beta$ and $\alpha^3\pi\ln(\beta)/80\beta$, respectively, in the velocity expansion above the threshold. As a consequence, $|G_m|^2$ and $|G_e|^2$ both contain the three-loop term $-\alpha^3\pi\ln(\beta)/3\beta$ in the velocity expansion. We would like to emphasize that the argument just given cannot be used to determine all three-loop contributions, but it clearly shows that a logarithmic Coulomb singularity also exists at order α^3/β which does not vanish in the limit $\beta\to 0$ in the cross section. The coefficient of this singularity further strongly implies that the $\ln(\beta)$ contributions in $|G_m|^2$ and $|G_e|^2$ to any number of loops and at NNLO in the velocity expansion above threshold can be cast into the factorized form

$$\Big[\,|G_{m/e}|^2\,\Big]_{\mathrm{NNLO}\ \ln\beta-\mathrm{contributions}} \sim \frac{z}{1-\exp\,(-z)}\,\Big(-\frac{2}{3}\,\alpha^2\,\ln\beta\Big)\,. \qquad (35)$$

It is clear from expression (35) and the arguments given above that the logarithmic Coulomb singularity does indeed affect the prediction for the cross section for $\beta\to 0$. In particular, we conclude that a conventional fixed order multi-loop calculation is not capable to determine NNLO (i.e. $\mathcal{O}(\alpha^2)$) relativistic corrections to the non-relativistic cross section[9]. In order to determine the correct form of the NNLO relativistic corrections to the non-relativistic cross section (or the Sommerfeld factor) resummations of the type mentioned before have to be performed. Such a program is beyond the scope of this work and will be carried out elsewhere.

[9] In a recent publication where large-n QCD sum rules were applied to the $b\bar{b}$ system [2] it was claimed that $\mathcal{O}(\alpha_s^2)$ accuracy was achieved in the determination of the strong coupling α_s and the bottom mass because two-loop corrections to the cross section were taken into account. Because the large-n limit peels out the threshold behavior of the $b\bar{b}$ production cross section, the results presented in [2] do not include NNLO relativistic effects properly and, therefore, are not at the $\mathcal{O}(\alpha_s^2)$ accuracy level. (See also [21].)

5 Summary

In this work we have determined the two-loop contributions to the electromagnetic form factors in the kinematic regime close to the fermion-antifermion threshold up to NNLO in an expansion in the velocity of the fermions in the c.m. frame. In the framework of NRQCD and NRQED the results are an important input for the two-loop renormalization of the effective Lagrangian. As the main outcome of this work we have demonstrated the existence of a new logarithmic (in the velocity) Coulomb singularity at NNLO in the velocity expansion. This logarithmic contribution belongs to the fermion-antifermion wave function and exists for the production of free fermion-antifermion pairs above threshold as well as for fermion-antifermion pairs in a bound state. For the case of fermion-antifermion pair production in e^+e^- annihilation the logarithm indicates that a resummation of contributions to any number of loops is mandatory in order to arrive at a viable (i.e. finite) prediction for the cross section with NNLO accuracy very close to the threshold point.

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