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Threshold behavior of Feynman diagrams: the master two-loop propagator

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Abstract

An asymptotic expansion of the two-loop two-point "master" diagram with two masses m and M, on the mass shell $Q^2 = M^2$, is presented. The treatment of the non-analytical terms arising in the expansion around the branching point is discussed. Some details of the calculation of a new class of two-loop integrals are given.

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1 Introduction

Phenomenological problems in particle physics often require calculations of multi-loop Feynman diagrams involving fields of various masses. Since exact results for diagrams with more than one mass scale practically do not exist beyond one loop, it is reasonable to apply some analytical approximations. In particular, one successfully applies explicit formulae for asymptotic expansions in various limits of momenta and masses [1] (see [2] for an informal review). The underlying idea is to utilize a hierarchy of mass scales in the given problem to reduce it to a calculation of one-scale diagrams. There are two general classes of problems which can be solved using those expansions: the large momentum case (when some external momenta are much larger than other relevant mass scales) and the case of very heavy virtual particles. In both classes important phenomenological problems have recently been solved by asymptotic expansions [3]. See also [4] where propagator and vertex two-loop diagrams were systematically calculated in various regions of momenta and masses.

There is, however, an important class of diagrams with a heavy particle both inside and on its mass shell on the external legs, when off-shell formulae [1] are generally not applicable[3]. Explicit formulae for asymptotic expansions in momenta and masses in some typical limits with large on-shell momenta were presented in [5]. In our paper we discuss the two-point on-shell integrals with two mass scales shown in Fig. 1 ($m < M$).

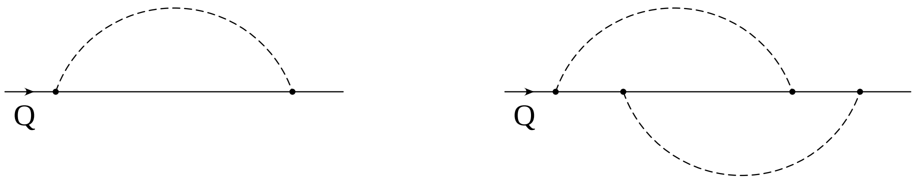

Figure 1: Examples of two-point on-shell ($Q^2 = M^2$) diagrams with two mass scales; solid lines denote a heavy particle with mass M, and dashed lines – a light one with mass m.

The one-loop diagram in Fig. 1a is a good example which illustrates both the difficulties of the previous approaches and the idea of the new method. Of course the exact result is well known:

$$
\begin{aligned}
G_1 &= \int \frac{\mathrm{d}^D k}{(k^2 - m^2)[(k+Q)^2 - M^2]} \\
&= i\pi^{D/2} \left(\frac{1}{\varepsilon} + 2 - \ln M^2 - x^2 \ln x - 2x\sqrt{4 - x^2}\, \arctan \sqrt{\frac{2-x}{2+x}} \right) \qquad (1)
\end{aligned}
$$

[3]Although these formulae are guaranteed at least off the mass shell, they are also valid in some on-shell situations, for example, in the pure large mass limit.

with $x = m/M$. In this paper we use dimensional regularization with $D = 4 - 2\varepsilon$; it is understood that the masses in propagators have an infinitesimal negative imaginary part which is not displayed. We drop Euler's γ in the formulae.

Suppose now we want to find the expansion of G_1 around $x = 0$ without calculating the above two-scale integral. Unfortunately it is impossible to apply just the Taylor expansion of the *integrand* around $m = 0$; the expanded propagator $\frac{1}{k^2-m^2} = \frac{1}{k^2}\sum_{n=0}^{\infty}\left(\frac{m^2}{k^2}\right)^n$ leads to infrared divergences. This difficulty is connected with the impossibility of the naive change of order of expansion and integration. General formulae of asymptotic expansions give, for this diagram, *two* contributions: besides this 'naive' Taylor expansion in m one has a contribution which is the Taylor expansion of another factor of the integrand, $\frac{1}{k^2+2kQ}$, around $k^2 = 0$ [5]. (In the off-shell situation, this extra contribution would be [1] just the Taylor expansion of $\frac{1}{(k+Q)^2-M^2}$ around $k = 0$, which happens to be impossible now because Q is *on-shell*, $Q^2 = M^2$.)

We therefore expand $\frac{1}{k^2+2kQ}$ around $k^2 = 0$ and obtain $\frac{1}{k^2+2kQ} = \frac{1}{2kQ}\sum_{n=0}^{\infty}\left(-\frac{k^2}{2kQ}\right)^n$. The resulting integrals are homogeneous in Q and depend on one mass scale m only. It is easy to convince oneself that in the result the divergences generated in the Taylor expansions are canceled and the sum of both contributions leads to

$$G_1 = i\pi^{D/2}\left[\frac{1}{\varepsilon} + 2 - \ln M^2 - \pi x + x^2(1-\ln x) + \frac{\pi}{8}x^3 + \mathcal{O}(x^4)\right]. \tag{2}$$

We notice that the terms $\pi\left(\frac{m}{M}\right)^{2n+1}$ appear even though m is found in the integrand only in an even power. These non-analytical terms arise because we expand the diagram around the point $m = 0$ which is also a branching point (threshold) of this Feynman diagram.

For the two-loop diagram in Fig. 1b no exact formula is known. In this paper we give its asymptotic expansion in the limit $m/M \to 0$. The next section is rather formal; it describes the general formalism of the expansion. In section 3 this machinery is applied to the diagram 1b; it turns out that there are four contributions (rather than two, as in the one-loop case). The calculation of a new type of integrals which arises in this context is discussed in some detail.

2 Explicit formula of asymptotic expansion

We consider the Feynman integral F_Γ corresponding to a graph Γ when the masses M_i and external momenta Q_i are large with the respect to small masses m_i and external momenta q_i. Let us suppose that the momenta are non-exceptional. Let the large external momenta be on the mass shell, $Q_i^2 = M_i^2$. Then the asymptotic expansion in the limit $Q_i, M_i \to \infty$ takes the following explicit form [5]:

$$F_\Gamma(Q_i, M_i, q_i, m_i; \varepsilon) \overset{M_i\to\infty}{\sim} \sum_\gamma \mathcal{M}_\gamma F_\Gamma(Q_i, M_i, q_i, m_i; \varepsilon). \tag{3}$$

Here the sum is over subgraphs γ of Γ such that

(a) in γ there is a path between any pair of external vertices associated with the large external momenta Q_i;

(b) γ contains all the lines with the large masses;

(c) every connectivity component γ_j of the graph $\hat{\gamma}$ obtained from γ by collapsing all the external vertices with the large external momenta to a point is 1PI with respect to the lines with the small masses.

In general γ can be disconnected. One can distinguish the connectivity component γ_0 which contains the external vertices with the large momenta.

The operator $\mathcal{M}_\gamma$ that is involved in the sum is a product $\prod_i \mathcal{M}_{\gamma_i}$ of operators of Taylor expansion in certain momenta and masses. These operators are by definition applied to integrands of Feynman integrals over loop momenta. For connectivity components γ_i other than γ_0 the corresponding operator performs Taylor expansion of the Feynman integral F_{γ_i} in its small masses and external momenta. (Note that its small external momenta are generally not only the small external momenta of the original Feynman integral but also some loop momenta of Γ.) Consider now $\mathcal{M}_{\gamma_0}$. The component γ_0 can be naturally represented as a union of its 1PI components and cut lines (after a cut line is removed the subgraph becomes disconnected; here they are of course lines with the large masses). By definition $\mathcal{M}_{\gamma_0}$ is again factorized and the Taylor expansion of the 1PI components of γ_0 is performed as in the case of other connectivity components γ_i.

It suffices now to describe the action of the operator $\mathcal{M}$ on the cut lines. Let l be such a line, with a large mass M_i, and let its momentum be $P_l + k_l$ where P_l is a linear combination of the large external momenta and k_l is a linear combination of the loop momenta and small external momenta. If $P_l = Q_i$ then the operator $\mathcal{M}$ for this component of γ is

$$\mathcal{T}_\kappa \frac{1}{\kappa k_l^2 + 2Q_i k_l}\bigg|_{\kappa=1} . \tag{4}$$

By $\mathcal{T}_x$ we denote the operator of the Taylor expansion in x around $x = 0$. In all other cases, e.g. when $P_l = 0$, or it is a sum of two or more external momenta, the operator $\mathcal{M}$ reduces to the ordinary Taylor expansion in small (with respect to this line considered as a subgraph) external momenta, i.e.

$$\mathcal{T}_k \frac{1}{(k_l + P_l)^2 - M_i^2} \equiv \mathcal{T}_\kappa \frac{1}{(\kappa k_l + P_l)^2 - M_i^2}\bigg|_{\kappa=1} . \tag{5}$$

In all cases apart from the cut lines with $P_l^2 = M_i^2$ the action of the corresponding operator $\mathcal{M}$ is graphically described by contraction of the corresponding subgraph to a point and insertion of the resulting polynomial into the reduced vertex of the reduced graph.

3 The two-loop master diagram

Let Γ be the two-loop self-energy graph (see Fig. 1b) containing three heavy lines with the mass M, two light lines with the mass m and let the external momentum be on the mass shell, $Q^2 = M^2$:

$$G_2 = \iint \frac{\mathrm{d}^D k \mathrm{d}^D l}{(k^2 - m^2)(l^2 - m^2)(k^2 + 2Qk)\left[(k+l)^2 + 2Q(k+l)\right](l^2 + 2Ql)}. \tag{6}$$

According to the general formula (3) four subgraphs contribute to the asymptotic expansion in the limit $m/M \to 0$:

(i) the graph Γ itself;

(ii) the subgraph γ_1 consisting of the left triangle and the third heavy line as well as symmetric subgraph γ_2;

(iii) the subgraph γ_3 consisting of three heavy lines.

For (i) we expand the integrand around $m = 0$:

$$\iint \frac{\mathrm{d}^D k \mathrm{d}^D l}{(k^2 + 2Qk)((k+l)^2 + 2Q(k+l))(l^2 + 2Ql)} \mathcal{T}_{m^2} \frac{1}{(k^2 - m^2)(l^2 - m^2)} \equiv \sum_{n=0}^{\infty} m^{2n} \sum_{j=0}^{n} I(j, n-j)\,. \tag{7}$$

where

$$I(a,b) = \iint \frac{\mathrm{d}^D k \mathrm{d}^D l}{(k^2)^{a+1}(l^2)^{b+1}(k^2 + 2Qk)\left[(k+l)^2 + 2Q(k+l)\right](l^2 + 2Ql)}. \tag{8}$$

The leading term (which gives G_2 at $m = 0$) is equal to [7]

$$I(0,0) = -\frac{\pi^D}{M^2}\left[\frac{3}{2}\zeta(3) - \pi^2 \ln 2\right] + \mathcal{O}(\varepsilon). \tag{9}$$

For any a and b the integral $I(a,b)$ can be reduced to $I(0,0)$ and one-loop integrals using recurrence relations [6, 8]. This algorithm has been implemented [9] in FORM [10].

Each of the two contributions to (ii) equals

$$\int \frac{\mathrm{d}^D l}{l^2 - m^2} \mathcal{T}_{l^2} \frac{1}{l^2 + 2Ql} \int \frac{\mathrm{d}^D k}{k^2 + 2Qk} \mathcal{T}_{m^2} \frac{1}{k^2 - m^2} \mathcal{T}_l \frac{1}{(k+l)^2 + 2Q(k+l)} \tag{10}$$

$$= \sum_{j_1, j_2, j_3 = 0}^{\infty} (-1)^{j_1} m^{2(j_1 + j_2)} \sum_{n=0}^{j_3} \binom{j_3}{n} \times \int \mathrm{d}^D l \frac{(2Ql + m^2)^{j_3 - n}}{(l^2 - m^2)(2Ql)^{j_1 + 1}} \int \mathrm{d}^D k \frac{(2kl)^n}{(k^2)^{j_2 + 1}(k^2 + 2Qk)^{j_3 + 2}}. \tag{11}$$

The factors l^2 in the numerator have been substituted by m^2 because whenever $l^2 - m^2$ appears it cancels the corresponding factor in the denominator and we get a massless tadpole integral which vanishes. The calculation of both one-loop integrals in (11) poses no difficulties.

Finally, the contribution (iii) is

$$\iint \frac{\mathrm{d}^D k \mathrm{d}^D l}{(k^2 - m^2)(l^2 - m^2)} \mathcal{T}_\kappa \frac{1}{(\kappa k^2 + 2Qk)(\kappa(k+l)^2 + 2Q(k+l))(\kappa l^2 + 2Ql)}\bigg|_{\kappa=1}$$
$$\equiv \sum_{j_1,j_2,j_3=0}^{\infty} \iint \mathrm{d}^D k \mathrm{d}^D l \frac{(-m^2)^{j_1+j_2}\left[-(k+l)^2\right]^{j_3}}{(k^2-m^2)(2Qk)^{j_1+1}(2Q(k+l))^{j_3+1}(l^2-m^2)(2Ql)^{j_2+1}}. \quad (12)$$

In this formula the following new type of integrals arises:

$$J(a_1, a_2, a_3, a_4, a_5) =$$
$$\iint \mathrm{d}^D k \mathrm{d}^D l \frac{(kl)^{a_5}}{(k^2 - m^2 + i0)^{a_1}(l^2 - m^2 + i0)^{a_2}(2Qk + i0)^{a_3}(2Qk + 2Ql + i0)^{a_4}}. \quad (13)$$

In general we can also have a power of $2Ql$ as an extra term in the denominator but this can be removed by partial fraction decomposition. Any such integral can be calculated analytically. We are interested in the case $a_1 = a_2 = 1$. In calculating (13) we first reduce the problem to the case $a_5 = 0$ by expressing the product $(kl)^{a_5}$ in terms of traceless products $(kl)^{(i)} \equiv k^{(\alpha,i)} l_{(\alpha,i)}$. Then we notice that the result of the integration over l of a traceless product $l_{(\alpha,i)}$ times the part of the integrand which depends only on l must be proportional to the traceless product $Q_{(\alpha,i)}$. Therefore the factor $(kl)^{(i)}$ can be replaced by $(Qk)^{(i)}(Ql)^{(i)}/(QQ)^{(i)}$. Then the factors involved are expressed through ordinary products Qk and Ql.

Now, an arbitrary integral $J(a_1, a_2, a_3, a_4, 0)$ is reduced to integrals with $a_3 = 0$ with the help of the following relation obtained by integration by parts

$$(2D - 2 - 2a_4 - a_3)\mathbf{3}^- - (D - 1 - a_4)\mathbf{4}^- + 4a_3\mathbf{3}^+ = 0. \quad (14)$$

In the present problem we only need the integrals J with the unit values of the first two arguments $a_1 = a_2 = 1$. In general it is also easy to reduce $a_{1,2}$ to 1 using a similar recurrence relation.

The relation (14) can be used to express an arbitrary integral $J(1, 1, a_3, a_4, 0)$ through integrals with $a_3 = 1$ and $a_3 = 0$. The latter will be considered below. To calculate $J(1, 1, 1, a_4, 0)$ we use the symmetry with respect to $l \leftrightarrow k$ and replace $2Ql$ in the numerator by $Qk + Ql$, which decreases a_4. In the resulting integral we use partial fraction decomposition with respect to $2Ql$. As a result we get integrals with decreased a_4 and products of one-loop integrals. If a_3 becomes increased in this process we apply the relation (14) again.

In the result of manipulations described above, the only two-loop integrals we are left with are of the form $J(1,1,0,a,0) \equiv J(a)$

$$J(a) = \iint \mathrm{d}^D k \mathrm{d}^D l \frac{1}{(k^2 - m^2 + i0)(l^2 - m^2 + i0)(2Qk + 2Ql + i0)^a} . \tag{15}$$

In $J(a)$ we first perform the k_0 and l_0 integrations using Cauchy theorem; the angular integrations are trivial since there are no products kl in the integrand. The two remaining radial integrations lead to

$$J(a) = -\frac{C^2 2^{1-2\varepsilon}}{(-4Mm)^a} \frac{\cos\left(\pi\varepsilon + \frac{\pi a}{2}\right)}{\cos(\pi\varepsilon)} B\left(\frac{3}{2} - \varepsilon, \frac{a}{2} - 1 + \varepsilon\right) B\left(\frac{a}{2} - 2 + 2\varepsilon, -\frac{a}{2} + \frac{3}{2} - \varepsilon\right) \tag{16}$$

with $C = \frac{2m^{2-2\varepsilon}\pi^{\frac{5}{2}-\varepsilon}}{\Gamma(\frac{3}{2}-\varepsilon)}$. This formula is valid to all orders in ε.

4 The final result

After adding the contributions (i), (ii), and (iii) we obtain the following finite result

$$G_2 = \frac{\pi^4}{M^2} \left\{ \pi^2 \ln 2 - \frac{3}{2}\zeta(3) + \sum_{n=1}^{18} \left[a_n + \pi^2 b_n + \ln(x^2) c_n \right] x^n + \mathcal{O}(x^{19}) \right\} . \tag{17}$$

The coefficients a_n and c_n are given in Table 1. For the coefficients of π^2 we find $b_{2n+1} = 0$, $b_{2n} = -\frac{1}{n 2^n}$. If this is true for all n this part of the series can be summed up analytically: $\pi^2 \sum_{n=1}^{\infty} b_{2n} x^{2n} = \pi^2 \ln(1 - x^2/2)$.

Similarly to the one-loop result (2) we see in the expansion (17) non-analytical terms with odd powers of x; at two-loop we also obtain logarithmic terms $\pi \left(\frac{m}{M}\right)^{2n+1} \ln\left(\frac{m}{M}\right)$.

The expansion (17) converges at least up to $m/M = 1$. We have compared the values obtained with this formula with the numerical program [11][4]. For $x < 0.5$ the accuracy is better than 10^{-5}; even at $x = 1$ the error is only 0.5%. Much more important than these numerical achievements is the possibility of extending the expansion (17) to any order in x. The methods described in this paper can also be extended to other cases of propagators and to vertex diagrams.

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[4]We thank J. Franzkowski for sending us an independent evaluation of G_2 at $x = 1$.

n	a_n	c_n
1	0	π
2	$-1/2$	0
3	$2\pi/3$	$\pi/24$
4	$-5/144$	$-1/12$
5	$13\pi/60$	$3\pi/640$
6	$107/5400$	$-1/15$
7	$529\pi/6720$	$5\pi/7168$
8	$9073/470400$	$-73/1680$
9	$14887\pi/483840$	$35\pi/294912$
10	$101923/7938000$	$-17/630$
11	$715801\pi/56770560$	$63\pi/2883584$
12	$3620783/461039040$	$-1391/83160$
13	$31515089\pi/5904138240$	$231\pi/54525952$
14	$47658179/10100170080$	$-317/30030$
15	$3278369671\pi/1416993177600$	$143\pi/167772160$
16	$24219791/8480609280$	$-19741/2882880$
17	$13114766971\pi/12847404810240$	$6435\pi/36507222016$
18	$8503364437/4825201661280$	$-21071/4594590$

Table 1: Coefficients of the expansion of the diagram G_2

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