HIGHER MOMENTS OF HEAVY QUARK VACUUM POLARIZATION

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We present analytical calculation of the first seven moments of the heavy quark vacuum polarization function in the three-loop order. The obtained results are compared against the asymptotic formulas following from the threshold singularities. We also discuss the μ dependence of the moments within the BLM procedure.

1 Introduction

The quark vacuum polarization function

$$\Pi^f(q^2) = \frac{-i}{3q^2}\int \mathrm{d}x e^{iqx}\langle 0|\; T\; j_\mu^f(x) j_\mu^f(0)\;|0\rangle, \tag{1}$$

with $j_\mu^f = \overline{\psi}_f\gamma_\mu\psi_f$ being the vector current for a quark f, is an interesting object related to a number of important physical quantities. An (incomplete) list includes:

- The combination[a]

$$D(q) = \frac{1}{q^2}\frac{1}{1+e^2\Pi^{em}(q^2)} \quad \text{with} \quad \Pi^{em} = \sum_f Q_f^2 \Pi^f \tag{2}$$

  is the quark contribution (to order e^2) to the photon propagator $D(q^2)$.

- The optical theorem relates the inclusive cross-section of e^+e^- annihilation into hadrons and thus the function $R(s) = \sigma_{tot}/\sigma_{point}$ to the discontinuity of Π^{em} in the complex plane

$$R(s) = 12\pi \,\mathrm{Im}\,\Pi^{em}(s+i\epsilon)\,. \tag{3}$$

[a] We ignore the so-called singlet contribution proportional to $(\sum_f Q_f)^2$ and first appearing in order α_s^3.

Conversely, the vacuum polarization is obtained through a dispersion relation from its absorptive part, vis.,

$$\Pi^{em}(q^2) \quad = \quad \frac{q^2}{12\pi^2}\int_{4m^2}^{\infty} ds\, \frac{R(s)}{s(s-q^2)}. \tag{4}$$

- Replacing the electromagnetic charge Q_f in Π^{em} by the vector weak coupling $v_f = 2I_3^f - 4Q_f \sin^2\theta_{\rm w}$ one arrives at the vector part of the quark contribution to the Z boson propagator and, again through the dispersion relation, to the vector part of the decay rate of the Z boson to hadrons.

To order α_s the calculation of Π^f was performed by Källén and Sabry in the context of QED a long time ago [1]. With measurements of ever increasing precision, predictions in order α_s^2 are needed for a reliable comparison between theory and experiment.

Such a calculation was recently presented in [2]. It employed a semianalytic approach based on conformal mapping and Padé approximation [3,4,5,6]. Important ingredients were the leading terms of the expansion of Π^f in the high energy region $(-q^2)/m^2 \gg 1$ [7,8], the Taylor series around $q^2 = 0$ which has been evaluated up to terms of order $(q^2)^4$ and information about the structure of Π in the threshold region. The method was tested in the case of the "double bubble" diagrams against the known analytical result [9] with highly satisfactory results. The calculation leads to a point-wise prediction of the absorptive part $R^f(s)$ in order α_s^2 with conservatively estimated accuracy of about 6%, which should be sufficient for comparison with experimental results in the foreseeable future.

An important case when the full mass dependence of Π^f is necessary is a precise determination of α_s and m_b from QCD sum rules for $\overline{b}b$ [10]. This is because of the still persisting contradiction between the values of α_s determined from low- and high-energy measurements [11]. Fortunately enough, QCD sum rules for $\overline{b}b$ are based on the moments of production cross section of $\overline{b}b$ states in e^+e^- annihilation which can been computed exactly even in order α_s^2.

In the present paper we report on analytical calculation of the first seven moments of the function Π^f or, equivalently, of its Taylor series around $q^2 = 0$ up to (and including) terms of order $(q^2)^7$.

2 Notations

We deal with the case of QCD containing $n_l = N_f - 1$ massless quarks and a massive quark ψ_F with the (pole) mass m_F later referred to as m. The corresponding polarization function Π^F and the ratio R^F will be denoted as Π and the ratio R, respectively. It is convenient to define

$$
\begin{aligned}
\Pi(q^2) &= \Pi^{(0)} + \frac{\alpha_s(\mu^2)}{\pi} C_F \Pi^{(1)} \qquad (5)\\
&+ \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left[\overbrace{C_F^2 \Pi_A^{(2)} + C_A C_F \Pi_{NA}^{(2)} + C_F T n_l \Pi_l^{(2)} + C_F T \Pi_F^{(2)}}^{\Pi^{(2)}}\right].
\end{aligned}
$$

with $\alpha_s(\mu)$ being the $\overline{\mathrm{MS}}$ coupling constant. For every polarization function $\Pi_?^{(i)}$ with $i = 0, 1, 2$ and $? = A, NA, l, F$ (if $i = 2$) we define its moments $C_?^{(i)}$ as follows

$$
\Pi_?^{(i)}(q^2) = \frac{3}{16\pi^2} \sum_{n>0} C_{?,n}^{(i)} \left(\frac{q^2}{4m^2}\right)^n \qquad (6)
$$

or, equivalently,

$$
C_{?,n}^{(i)} = \frac{4}{9} \int_{4m^2}^{\infty} R_{?,n}^{(i)}(s) \left(\frac{4m^2}{s}\right)^n \frac{ds}{s} = \frac{4}{9} \int_0^1 R_{?,n}^{(i)}(v) \left(1 - v^2\right)^{n-1} d(v^2), \qquad (7)
$$

with $v = \sqrt{1 - \frac{4m^2}{s}}$. Note that by definition

$$
C_n^{(2)} = C_F^2 C_{A,n}^{(2)} + C_A C_F C_{NA,n}^{(2)} + C_F T n_l C_{l,n}^{(2)} + C_F T C_{F,n}^{(2)}. \qquad (8)
$$

In order to fix the absolute normalization we put below the lowest order result for $R(s)$:

$$
R^{(0)}(s) = 3\left(1 + \frac{2m^2}{s}\right)\sqrt{1 - \frac{4m^2}{s}}. \qquad (9)
$$

3 The calculation

Important information is contained in the Taylor series of $\Pi(q^2)$ around zero. The first seven coefficients of the Taylor series around $q^2 = 0$ are calculated with the help of the program MATAD which has been used previously for the three-loop corrections to the ρ parameter [12], to Δr [13] and the top loop induced reaction $e^+e^- \to Hl^+l^-$ [14]. The package is written in FORM [15]. For a given diagram it calculates the derivatives with respect to the external momentum q^2 to the desired order, raising internal propagators to powers up to about 20. It then performs the traces and uses recurrence relations based on the integration by parts method [16,17] to reduce the resulting three-loop tadpole diagrams (in dimensional regularisation). In order to save space we present our results converted in numerical form in the Table 1. The original analytical expressions can be found in our paper [18].

The coefficients $C_1^{(2)} - C_4^{(2)}$ were already presented in [2], $C_5^{(2)}, C_6^{(2)}$ and $C_7^{(2)}$ are new. The C_F^2 term of $C_1^{(2)} - C_3^{(2)}$ and $C_5^{(2)}$ are in agreement with [6] and [19], respectively. The results in the Table are given for a particular choice of $\mu = m$; in order to transform them to a general μ the following relation should be used:

$$C_n^{(2)}(\mu') = C_n^{(2)}(\mu) + C_n^{(1)}\beta_0 \ln\left(\frac{\mu'}{\mu}\right)^2 \quad \text{with} \quad \beta_0 = \frac{11}{12}C_A - \frac{T}{3}N_f \tag{10}$$

n	$C_n^{(0)}$	$C_n^{(1)}$	$C_{A,n}^{(2)}$	$C_{NA,n}^{(2)}$	$C_{l,n}^{(2)}$	$C_{F,n}^{(2)}$	$C_n^{(2),5}$	$C_n^{(2),4}$
1	1.067	4.049	5.075	7.098	-2.339	0.7270	31.66	33.22
2	0.4571	2.661	6.393	6.311	-2.174	0.2671	30.99	32.44
3	0.2709	2.015	6.689	5.398	-1.896	0.1499	28.53	29.79
4	0.1847	1.63	6.685	4.699	-1.671	0.0995	26.29	27.40
5	0.1364	1.372	6.574	4.165	-1.494	0.0723	24.41	25.41
6	0.1061	1.186	6.426	3.746	-1.353	0.0557	22.84	23.74
7	0.0856	1.046	6.267	3.409	-1.239	0.0446	21.5	22.33

Table 1: The results for the first seven moments. The last two columns display the moments $C_n^{(2)}$ for the cases of $N_f = 5$ and $N_f = 4$, respectively.

4 Higher n moments and threshold singularities

In this section we compare our exact moments with their asymptotic behaviour originating from the corresponding threshold singularities, with $r_{?,n}^{(i)}$ standing for the ratio $C_{?,n}^{(i)}/C_{?,n}^{(i),as}$.

As it follows from (7) the asymptotic behaviour of the moments in the limit of large n is completely determined by the behaviour of $R(s)$ at the threshold $s \approx 4m^2$. The constraints on the threshold behaviour of $\Pi(q^2)$ originate from our knowledge about the nonrelativistic Greens function in the presence of a Coulomb potential and its interplay with "hard" vertex corrections. For a theory with nonvanishing β function (QED with light fermions or QCD) the proper definition of the coupling constant and its running must be taken into account. Below we cite results that are available in the literature about the threshold behaviour of $R(s)$ and its moments.

$$R^{(0),thr} = 3\left(\frac{3v}{2} - \frac{v^3}{3}\right), \tag{11}$$

$$C_n^{(0),as} = \frac{\sqrt{\pi}}{n^{3/2}} - \frac{7\sqrt{\pi}}{8n^{5/2}}. \tag{12}$$

$$R^{(1),thr} = 3\left(\frac{3}{4}\pi^2 - 6v\right), \tag{13}$$

$$C_n^{(1),as} = \frac{\pi^2}{n} - \frac{4\sqrt{\pi}}{n^{3/2}}. \tag{14}$$

C_F^2 **part** [20,21,6]:

$$R_A^{(2),thr} = 3\left(\frac{\pi^4}{8v} - 3\pi^2\right), \tag{15}$$

$$C_{A,n}^{(2),as} = \frac{\pi^{9/2}}{6n^{1/2}} - 4\frac{\pi^2}{n}. \tag{16}$$

$C_F C_A$ **part** [2]:

$$R_{NA}^{(2),thr} = 3\pi^2\left(-\frac{11}{16}\ln\frac{v^2 s}{\mu^2} + \frac{31}{48}\right), \tag{17}$$

$$C_{NA,n}^{(2),as} = \frac{\pi^2}{12n}\left(\frac{31}{3} - 11\ln 4 + 11\sum_{i=1}^{n-1}\frac{1}{i}\right). \tag{18}$$

$C_F n_l T$ **part** [9]:

$$R_l^{(2),thr} = 3\pi^2\left(\frac{1}{4}\ln\frac{v^2 s}{\mu^2} - \frac{5}{12}\right), \tag{19}$$

$$C_{l,n}^{(2),as} = \frac{\pi^2}{3n}\left(-\frac{5}{3} + \ln 4 - \sum_{i=1}^{n-1}\frac{1}{i}\right). \tag{20}$$

$C_F T$ **part** [9]:

$$R_F^{(2),thr} = (22 - 2\pi^2)v + (-\frac{245}{18} + \frac{4}{3}\pi^2)v^3, \tag{21}$$

$$C_{F,n}^{(2),as} = \frac{\sqrt{\pi}}{9n^{3/2}} \left(-4\pi^2 + 44 - \frac{172}{3n} + \frac{11\pi^2}{2n} \right). \tag{22}$$

In Table 2 we compare the asymptotic formulas with the available moments. We observe that at $n = 7$ the first two terms of the asymptotic expansion in $1/n$ agree with the exact results with the accuracy of about 35%.

n	$r_n^{(0)}$	$r_n^{(1)}$	$r_{A,n}^{(2)}$	$r_{NA,n}^{(2)}$	$r_{l,n}^{(2)}$	$r_{F,n}^{(2)}$	$r_n^{(2),5}$	$r_n^{(2),4}$
1	4.814	1.457	-0.474	-1.755	2.536	2.51	-0.845	-0.902
2	1.297	1.096	10.51	2.522	1.032	1.28	5.522	4.624
3	1.121	1.046	1.937	1.700	0.9709	1.128	2.078	1.982
4	1.067	1.031	1.479	1.499	0.9611	1.075	1.643	1.597
5	1.043	1.024	1.322	1.407	0.9608	1.049	1.472	1.442
6	1.03	1.020	1.243	1.353	0.9628	1.035	1.38	1.357
7	1.022	1.018	1.197	1.317	0.9653	1.026	1.321	1.304

Table 2: The results for the first seven ratios of exact moments to their asymptotic values. The last two column display the ratios for the case of $N_f = 5$ and $N_f = 4$, respectively.

5 Choice of μ

The μ dependence of $C^{(2)}$ as displayed by eq. (10) is induced by the running of $\alpha_s(\mu^2)$. Now we want to apply the BLM procedure[22] to our results. It suggests to choose the scale of the $\mathcal{O}(\alpha_s)$ term in such a way that the contribution of the $\mathcal{O}(\alpha_s^2)$ part proportional to β_0 is absorbed. This prescription is based on the observation that the remaining coefficients of the α_s^2 terms are often relatively small. It is possible to treat each term of $\mathcal{O}(z^n)$ separately. In Table 3 we list the BLM scale μ_{BLM} for each Taylor coefficient together with the numerical value of the original ($C_n^{(2)}$) and the $\mathcal{O}(\alpha_s^2)$ term which remains after the BLM scale is adjusted (C_n^{BLM}). It is interesting to note that μ_{BLM} is decreasing with increasing n. This is plausible because the higher Taylor coefficients are increasingly dominated by the threshold region and the characteristic scale at threshold is the relative three-momentum of the quarks. Note that C_n^{BLM}

n	$C_n^{(0)}$	$C_n^{(1)}$	$C_n^{(2)}$	C_n^{BLM}	μ_{BLM}/m	$\bar{C}_n^{(1)}$	$\bar{C}_n^{(2)}$
1	1.07	4.05	30.1	13.7	0.420	1.92	3.82
2	0.46	2.66	29.5	14.3	0.294	0.83	3.69
3	0.27	2.01	27.3	14.0	0.244	0.39	2.50
4	0.18	1.63	25.2	13.5	0.215	0.15	1.65
5	0.14	1.37	23.4	13.0	0.195	0.01	1.12
6	0.11	1.19	21.9	12.5	0.181	-0.09	0.85
7	0.09	1.05	20.7	12.0	0.169	-0.15	0.76

Table 3: The BLM scale μ_{BLM} is expressed in terms of the original scale m. For the numerical values of $C_n^{(2)}$ and $\bar{C}_n^{(2)}$ $n_l = 5$ and $\mu = m$, respectively, $\mu = \bar{m}$ is used. The double bubble diagrams with two massive fermion loops are also included. C_n^{BLM} is the coefficient remaining after the β_0 term is absorbed.

remains nearly constant whereas $C_n^{(2)}$ decreases for increasing n (but remember the rapidly decreasing coefficients $C_n^{(0)}$).

In comparison to $C_n^{(2)}$ in Table 3 also the corresponding one- and two-loop coefficients $(C_n^{(0)}, C_n^{(1)})$ and the two- and three-loop coefficients $(\bar{C}_n^{(1)}, \bar{C}_n^{(2)})$ in the $\overline{\text{MS}}$ scheme are listed. Whereas $C_n^{(1)}$ is roughly of $\mathcal{O}(1)$ $\bar{C}_n^{(1)}$ varies from approximately 1.9 down to 0.01. Also the sign changes. Therefore the BLM procedure is rather unstable and not applicable. On the other hand, the $\overline{\text{MS}}$ coefficients $\bar{C}_n^{(2)}$ are already reasonably small, so there is no urgent need for an optimization procedure.

6 Conclusions and summary

We have presented analytical results for first seven moments of the heavy quark vacuum polarization function to order $\mathcal{O}(\alpha_s^2)$. We found that BLM procedure does not lead to a significant improvement of the perturbation theory for the moments. It is demonstrated that the threshold singularities do describe the higher moments (with $n = 6, 7$) with inaccuracy not exceeding 40% (in order α_s^2).

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